Problem 5.3.5

Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{(1-p)^{n-1}p}{n} & k=1,\ldots,n; \quad n=1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

→ k'ya göre uniform

Find the marginal PMFs $P_N(n)$ and $P_K(k)$.

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = \sum_{k=1}^{n} \frac{(1-p)^{n-1}p}{n}$$

$$P_N(n) = (1-p)^{n-1}p \qquad n = 1,2,\ldots$$

$N \sim \text{Geom}(p)$

$$P_K(k) = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n}$$

$$= p\sum_{n=1}^{\infty} \frac{A^{n-1}}{n} = \frac{p}{A}\sum_{n=1}^{\infty} \frac{A^n}{n}$$

$$p\frac{d}{dA}\left(\sum_{n=1}^{\infty} A^n\right) = p\sum_{n=1}^{\infty} n A^{n-1}$$

$$p\int\left(\sum_{n=1}^{\infty} A^{n-1}\right)dA = p\sum \frac{A^n}{n} = pA\sum_{n=1}^{\infty} \frac{A^{n-1}}{n}$$

$$\sum_{n=1}^{\infty} A^{n-1} = \frac{1}{1-A}$$

$$= \int \frac{p}{1-A}\,dA = \frac{p}{(1-A)^2} = \frac{p}{p^2} = \frac{1}{p}, \quad k = 1,2,\ldots,n$$

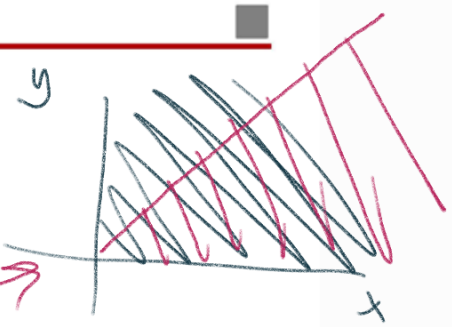
uniform

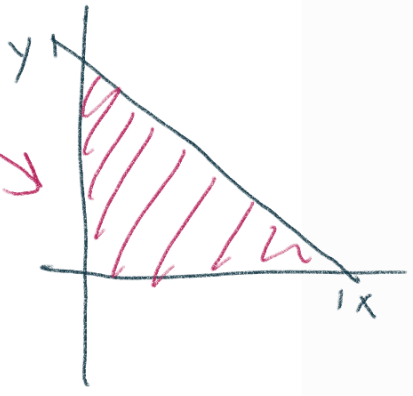
Problem 5.4.2

Random variables X and Y have joint PDF

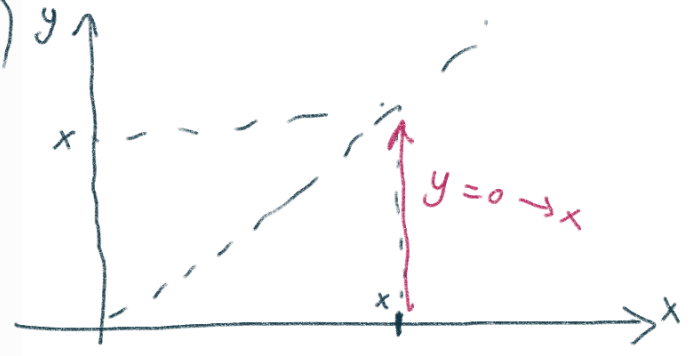
$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P[X > Y]$ and $P[X + Y \leq 1]$.
(b) Find $P[\min(X,Y) \geq 1]$.
(c) Find $P[\max(X,Y) \leq 1]$.





a)



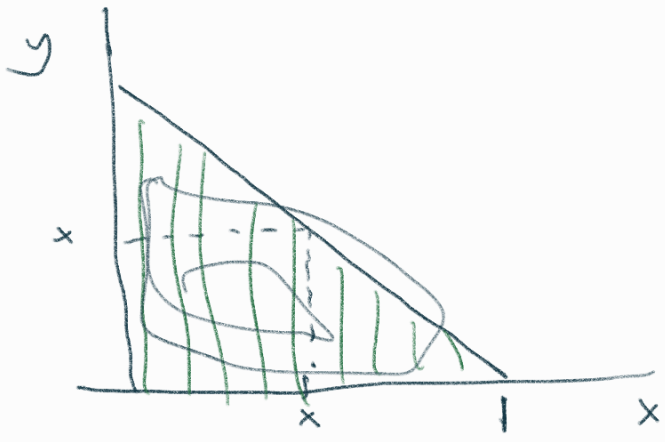
$$\int_{x=0}^{\infty} \int_{y=0}^{x} 6 e^{-2x} e^{-3y} \, dy \, dx$$

$$= \int_{x=0}^{\infty} \left(-2 e^{-3y} e^{-2x} \Big|_{y=0}^{x} \right) dx$$

$$-2e^{-5x} + 2e^{-2x}$$

$$= \int_{x=0}^{\infty} -2e^{-5x} + 2e^{-2x} \, dx$$

$$\frac{2}{5} e^{-5x} - e^{-2x} \Big|_{x=0}^{\infty} = -\frac{2}{5} + 1 = \frac{3}{5} //$$



$$\int_{x=0}^{1}\int_{y=0}^{x} 6e^{-2x}e^{-3y}\,dy\,dx$$

$$\int_{x=0}^{1}\left(-2e^{-2x}e^{-3y}\Big|_{y=0}^{x}\right)dx$$

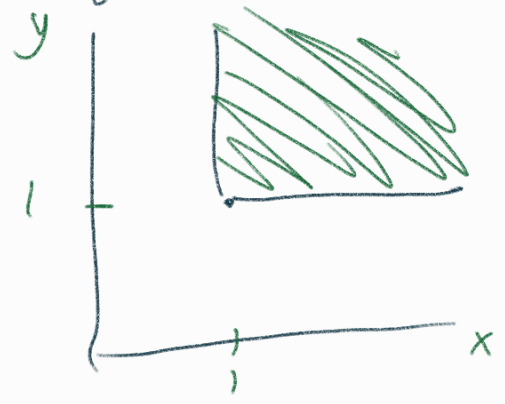
$$\int_{x=0}^{1}\left(-2e^{-5x}+2e^{-2x}\right)dx$$

$$= \frac{2}{5}e^{-5x} - e^{-2x}\Big|_{x=0}^{1} = \frac{2}{5}e^{-5} - e^{-2} - \frac{2}{5} + 1$$

$$= \frac{3}{5} + \frac{2}{5}e^{-5} - e^{-2}$$

b) $P[\min\{X,Y\} \geq 1]$

$P[X \geq 1, Y \geq 1]$



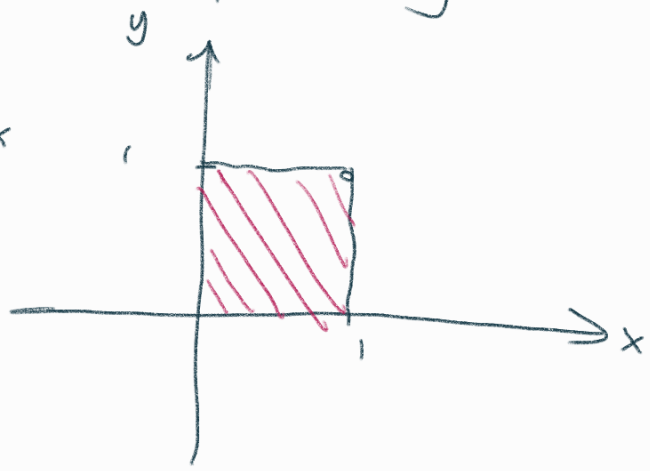
$$\int_{x=1}^{\infty}\int_{y=1}^{\infty} 6e^{-2x}e^{-3y}\,dy\,dx$$

$$\int_{x=1}^{\infty}\left(-2e^{-2x}e^{-3y}\Big|_{y=1}^{\infty}\right)dx$$

$$\int_{x=1}^{\infty} 2e^{-2x}e^{-3}\,dx = -e^{-2x}e^{-3}\Big|_{x=1}^{\infty} = e^{-5}$$

c) $P[\max(X,Y) \le 1] = P(X \le 1, Y \le 1)$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 6e^{-2x} e^{-3y} \, dy \, dx$$



$$= \int_{x=0}^{1} \left(-2e^{-2x} e^{-3y} \Big|_{y=0}^{1} \right) dx$$

$$= \int_{x=0}^{1} -2e^{-2x} e^{-3} + 2e^{-2x} \, dx$$

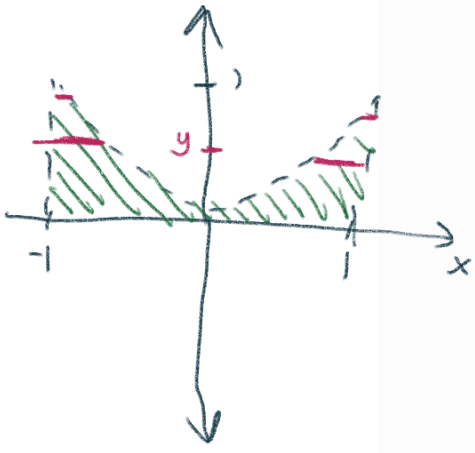
$$= e^{-2x} e^{-3} - e^{-2x} \Big|_0^1 = e^{-5} - e^{-2} - e^{-3} + 1$$

$$= 1 + e^{-5} - e^{-2} - e^{-3}$$

Problem 5.5.5

X and Y are random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1; \ 0 \le y \le x^2 \\ 0 & \text{otherwise.} \end{cases}$$



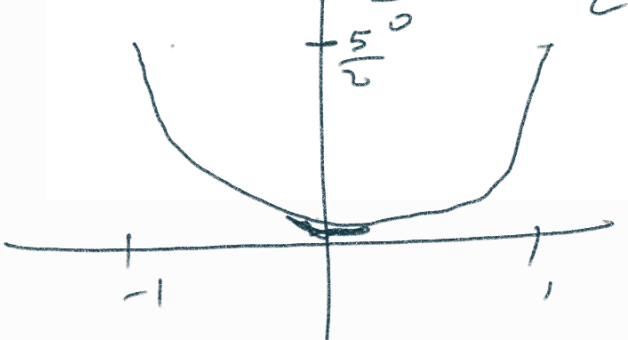
(a) What is the marginal PDF $f_X(x)$?

(b) What is the marginal PDF $f_Y(y)$?

a) $$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y) \, dy$$

$$= \int_{0}^{x^2} \frac{5x^2}{2} \, dy = \frac{5x^2 y}{2} \Big|_{y=0}^{x^2}$$

$$f_X(x) = \frac{5}{2} x^4, \quad -1 \le x \le 1$$



b) $f_Y(y) = \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\,dx + \int_{+\sqrt{y}}^{1} \frac{5x^2}{2}\,dx$

$= \frac{5x^3}{6}\Big|_{-1}^{-\sqrt{y}} + \frac{5x^3}{6}\Big|_{\sqrt{y}}^{1} = \frac{5}{6} - \frac{5y\sqrt{y}}{6}$

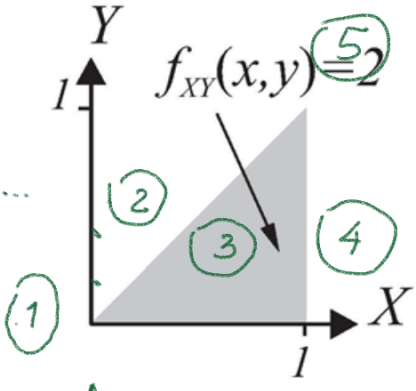
$= -\frac{5y\sqrt{y}}{6} + \frac{5}{6}$

$f_Y(y) = \frac{5}{3} - \frac{5}{3}y\sqrt{y} \qquad 0 \le y \le 1$

Example 5.8 Problem

$y \to v$
$x \to u$

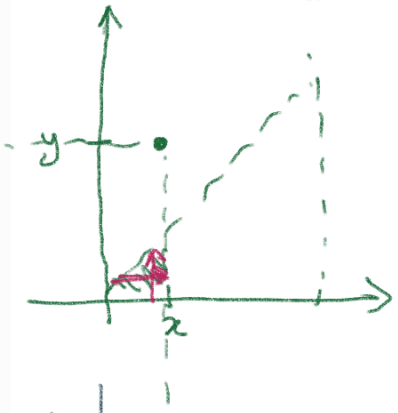
Find the joint CDF $F_{X,Y}(x,y)$ when X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \qquad (5.22)$$

(1) $x \le 0$ veya $y \le 0 \Rightarrow F_{X,Y}(x,y) = 0$

(2) $y \ge x$ ve $0 \le x \le 1$

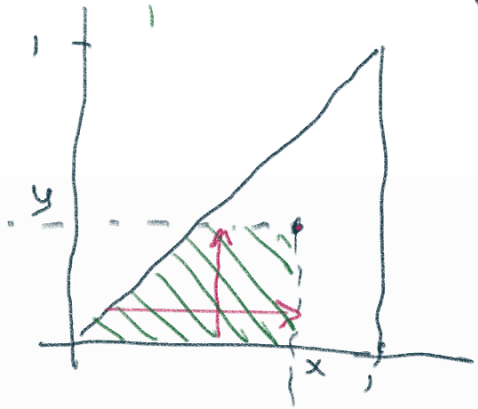


$$\int_{v=0}^{x}\int_{u=v}^{x} 2\,du\,dv = x^2$$

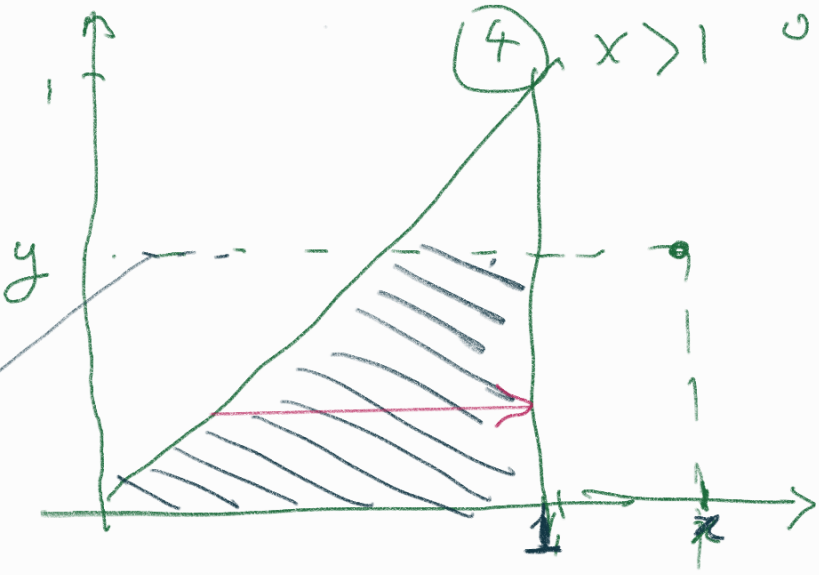
(3) $y < x$ ve $0 \le x \le 1$

$u = v \longrightarrow x$

$v = 0 \longrightarrow y$



$$\int_{v=0}^{y}\int_{u=v}^{x} 2\,du\,dv = 2xy - y^2$$

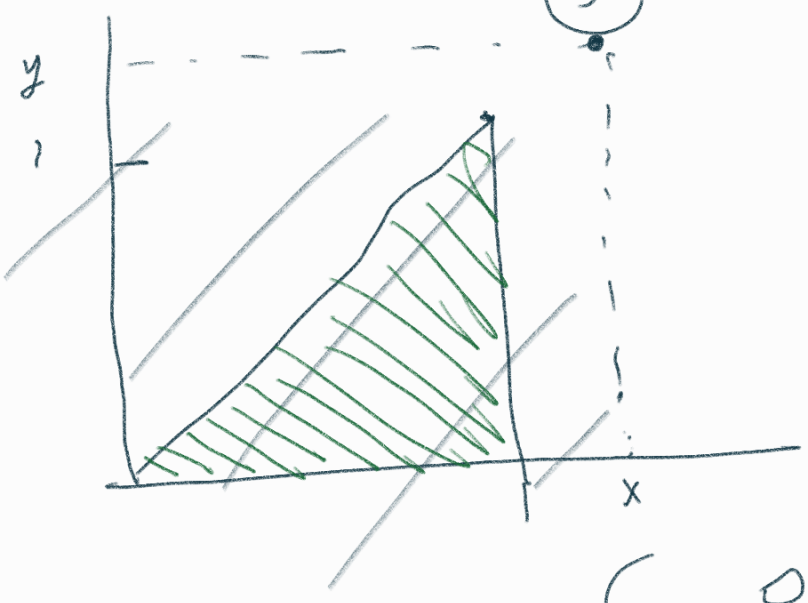


(4) $x>1$ $0\le y\le 1$

$u = v \to 1$

$v = 0 \to y$

$$\int_{v=0}^{y}\int_{u=v}^{1} 2\,du\,dv = 2y-y^2$$



(5) $x>1$, $y>1$

$F_{xy}(x,y)=1$

$$F_{xy}(x,y)=\begin{cases} 0 & x<0 \text{ veya } y<0 \\ x^2 & x\le y,\ 0\le x\le 1 \\ 2xy-y^2 & y<x,\ 0\le x<1 \\ 2y-y^2 & 1\le x,\ 0\le y\le 1 \\ 1 & x>1,\ y>1 \end{cases}$$

Problem 5.5.9

X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cy & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Draw the region of nonzero probability.
(b) What is the value of the constant c?
(c) What is $F_X(x)$?
(d) What is $F_Y(y)$?
(e) What is $P[Y \le X/2]$?

a)

b) $\iint_{S_{XY}} f_{XY}(x,y)\,dx\,dy = 1$ olmalı

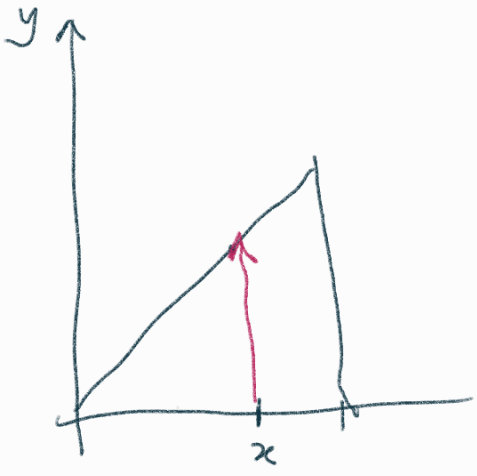
$$\int_{y=0}^{1}\int_{x=y}^{1} cy\,dx\,dy = \int_{y=0}^{1} \underbrace{\left(cyx\Big|_{x=y}^{1}\right)}_{cy-cy^2} dy$$

$$= \int_{y=0}^{1} cy - cy^2\,dy = \frac{cy^2}{2} - \frac{cy^3}{3}\Big|_0^1 = \frac{c}{2} - \frac{c}{3} = 1$$

$$\Rightarrow c = 6$$

c) $F_X(x) = ?$

önce $f_X(x)$'i bulalım.



$$f_X(x) = \int f_{XY}(x,y)\,dy$$

$$f_X(x) = \int_{y=0}^{x} 6y\,dy \qquad , 0 \le x \le 1$$

$$= 3y^2 \Big|_{y=0}^{x} = 3x^2 \Rightarrow \boxed{f_X(x) = 3x^2, \quad 0 \le x \le 1}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{0}^{x} 3u^2\,du = u^3 \Big|_{0}^{x} = x^3$$

d) $F_Y(y) = \int_{0}^{y} f_Y(y)\,dy$

$f_Y(y)$?

$$f_Y(y) = \int_{y}^{1} f_{XY}(x,y)\,dx \qquad X = y \to 1$$

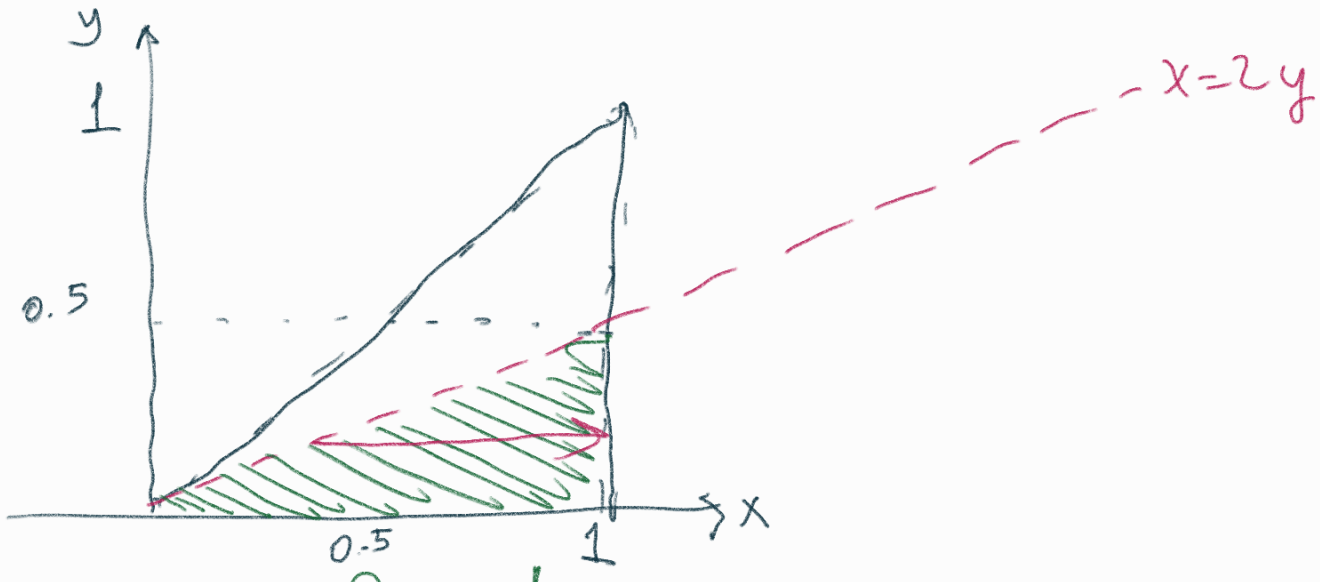
$$= \int_{x=y}^{1} 6y\,dx = 6yx \Big|_{x=y}^{x=1} = 6y - 6y^2 \qquad 0 \le y \le 1$$

$$F_Y(y) = \int_{0}^{y} 6v - 6v^2\,dv = 3v^2 - 2v^3 \Big|_{v=0}^{y}$$

$$\boxed{F_Y(y) = \begin{cases} 0 & y \le 0 \\ 3y^2 - 2y^3 & 0 \le y \le 1 \\ 1 & 1 < y \end{cases}}$$

e) $P(Y < X/2)$

$X > 2Y$



$$P(Y<X/2) = \int_{y=0}^{} \int_{X=2y}^{1} 6y \, dx \, dy$$

$$\int_{y=0}^{0.5} \left(6y \times \Big|_{x=2y}^{x=1} \right) dy = \int_{y=0}^{0.5} 6y - 12y^2 \, dy$$

$$3y^2 - 4y^3 \Big|_{y=0}^{0.5}$$

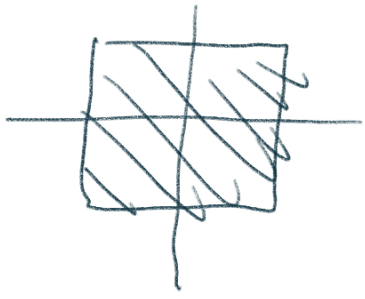
$$= \frac{3}{4} - \frac{4}{8} = \frac{2}{8} = \frac{1}{4} //$$

Problem 5.6.7

In terms of a positive constant k, random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k+3x^2 & -1/2 \le x \le 1/2, \\ & -1/2 \le y \le 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is k?
(b) What is the marginal PDF of X?
(c) What is the marginal PDF of Y?
(d) Are X and Y independent?



$$1 = \int_{y=-1/2}^{1/2} \int_{x=-1/2}^{1/2} (k+3x^2)\,dx\,dy$$

$$\int_{y=-1/2}^{1/2} \left(kx + x^3 \Big|_{x=-1/2}^{x=1/2} \right) dy$$

$$\int_{y=-1/2}^{1/2} \left(k + \frac{1}{4} \right) dy = ky + \frac{y}{4} \Big|_{-1/2}^{1/2} = k + \frac{1}{4} = 1$$

$$k = 3/4$$

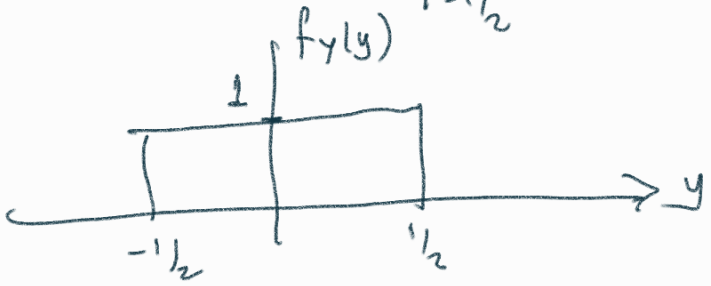
b) $$f_X(x) = \int_{y=-1/2}^{1/2} f_{XY}(x,y)\,dy$$

$$\int_{y=-1/2}^{1/2} \frac{3}{4} + 3x^2\,dy = \frac{3y}{4} + 3x^2 y \Big|_{y=-1/2}^{y=1/2}$$

$$f_X(x) = \frac{3}{4} + 3x^2, \quad -\frac{1}{2} \le x \le 1/2$$

c) $$f_Y(y) = \int_{x=-1/2}^{1/2} f_{XY}(x,y)\,dx = \int_{x=-1/2}^{1/2} \frac{3}{4} + 3x^2\,dx$$

$= \frac{3}{4}x + x^3 \Big|_{-1/2}^{1/2} = \frac{3}{4} + \frac{1}{4} = 1 \quad , -\frac{1}{2} \le y \le 1/2$



$f_X(x) f_Y(y) = 1 \times \left(\frac{3}{4} + 3x^2\right) \stackrel{?}{=} f_{XY}(x,y)$

$\to \frac{3}{4} + 3x^2$

EVET, EŞİTTİR
YANİ BAĞIMSIZDIR

Problem 5.6.8

X_1 and X_2 are independent, identically distributed random variables with PDF

$$f_X(x) = \begin{cases} x/2 & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the CDF, $F_X(x)$.

(b) What is $P[X_1 \le 1, X_2 \le 1]$, the probability that X_1 and X_2 are both less than or equal to 1?

(c) Let $W = \max(X_1, X_2)$. What is $F_W(1)$, the CDF of W evaluated at $w = 1$?

(d) Find the CDF $F_W(w)$.

a) $F_X(x) = \int_{u=0}^{x} f_X(u)\,du = \int_{u=0}^{x} \frac{u}{2}\,du = \frac{u^2}{4}\Big|_{u=0}^{x}$

$$f_X(x) = \begin{cases} \frac{x^2}{4}, & 0 \le x \le 2 \\ 1 & 2 < x \\ 0 & x < 0 \end{cases}$$

b) $P[X_1 \le 1, X_2 \le 1]$

$= P[X_1 \le 1]\, P[X_2 \le 1]$

$= F_{X_1}(1)\, F_{X_2}(1) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

c) $W = \max(X_1, X_2)$

$$F_W(1) = P(W \le 1) = P(\max(X_1, X_2) \le 1)$$
$$= P(X_1 \le 1, X_2 \le 1)$$
$$= P(X_1 \le 1)\, P(X_2 \le 1)$$
$$= F_{X_1}(1)\, F_{X_2}(1) = \frac{1}{16}$$ aynı soru

d) $F_W(w) = ?$

$$= P(W \le w)$$
$$= P(X_1 \le w) P(X_2 \le w)$$
$$= \frac{w}{2} \cdot \frac{w}{2} = \frac{w^2}{4}$$

$$F_W(w) = \begin{cases} 0 & w \le 0 \\ w^2/4 & 0 < w \le 2 \\ 1 & 2 < w \end{cases}$$

Problem 5.7.5

X and Y are random variables with $E[X] = E[Y] = 0$ and $\text{Var}[X] = 1$, $\text{Var}[Y] = 4$ and correlation coefficient $\rho = 1/2$. Find $\text{Var}[X + Y]$.

Covariance (Kovaryans) $= \text{Cov}(X,Y)$

$$= E[XY] - E[X]\,E[Y]$$

Correlation coefficient (Ilinti Katsayisi) $= \rho = \dfrac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$

$$-1 \le \rho \le 1$$

$$\rho = \frac{1}{2} = \frac{E[XY] - E[X]\,E[Y]}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{E[XY]}{2}$$

($\text{Var}(X) \to 1$, $\text{Var}(Y) \to 4$)

$$E[XY] = 1 \longrightarrow \text{Cov}(X,Y) = 1$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)$$

$$= E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - (E[Y])^2 - 2\,E[X]\,E[Y]$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)$$

$$\text{Var}(X+Y) = 1 + 4 + 2 = 7$$

$$E[(X-\mu_X)(Y-\mu_Y)] > 0$$

Sifr ilinti $\Rightarrow$ $E[XY] = E[X]E[Y]$

$\Downarrow$

$Cov(X,Y) = 0$

Problem 5.7.9

X and Y are identically distributed random variables with $E[X] = E[Y] = 0$ and covariance $\mathrm{Cov}[X,Y] = 3$ and correlation coefficient $\rho_{X,Y} = 1/2$. For nonzero constants a and b, $U = aX$ and $V = bY$.

(a) Find $\mathrm{Cov}[U,V]$.

(b) Find the correlation coefficient $\rho_{U,V}$.

(c) Let $W = U+V$. For what values of a and b are X and W uncorrelated?

a) $\mathrm{Cov}(U,V) = E[UV] - E[U]E[V]$

$= E[abXY] - E[aX]E[bY]$

$= ab\,E[XY] - ab\,E[X]E[Y]$

$= ab\,(E[XY] - E[X]E[Y])$

$= ab\,\mathrm{Cov}(X,Y)$ ($\mathrm{Cov}(X,Y) = 3$)

$= 3ab$ //

b) $\rho_{U,V} = \dfrac{Cov(U,V)}{\sqrt{Var(U)\,Var(V)}} = \dfrac{3ab}{\sqrt{a^2 Var(X)\; b^2 Var(Y)}}$

$= \dfrac{3 \to Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \rho_{X,Y} = \frac{1}{2}$

c) $W = U + V$

$W = aX + bY$

$E[XW] = E[X]\,E[W]$ $\Rightarrow$ hangi a ve b için?

yani;

$E[XW] = 0 \Rightarrow$ hangi a ve b için

$E[X(aX+bY)] = a\,E[X^2] + b\,E[XY]$

$Var(X) = 6$; $E[XY] = 3$

$= 6a + 3b = 0$ olursa

$\boxed{2a + b = 0}$ olursa

$Var(X) = ?$

$\dfrac{3}{\sqrt{Var(X)Var(Y)}} = \frac{1}{2}$

$6^2 = Var(X)\,Var(Y)$

$Var(X) = 6 \to E[X^2] = 6$

X ve W ilintisiz olur.

Problem 5.7.10

True or False: For identically distributed random variables Y_1 and Y_2 with $E[Y_1] = E[Y_2] = 0$, $\text{Var}[Y_1 + Y_2] \geq \text{Var}[Y_1]$.

$\rightarrow \text{Var}(Y_1+Y_2) = \text{Var}(Y_1) + \text{Var}(Y_2) + 2\,\text{Cov}(Y_1, Y_2)$

$\overset{?}{\geq} \text{Var}(Y_1)$

$\times \text{Cov}(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$

$\rightarrow \text{Var}(Y_2) + 2\,\text{Cov}(Y_1, Y_2) \overset{?}{\geq} 0$

$\rightarrow \text{Var}(Y_2) + 2\,\frac{\text{Cov}(Y_1,Y_2)}{\sqrt{\text{Var}(Y_1)}\sqrt{\text{Var}(Y_2)}}\sqrt{\text{Var}(Y_1)\,\text{Var}(Y_2)} \geq 0$

$\sigma^2 + 2\rho\sigma^2 \geq 0$

$1 + 2\rho \geq 0 \Rightarrow \rho \overset{?}{\geq} -\frac{1}{2}$

Demek ki $-1 \leq \rho < -\frac{1}{2}$ olursa o zaman bu esitlik her zaman dogru degil.

Problem 5.7.11

X and Y are random variables with $E[X] = E[Y] = 0$ such that X has standard deviation $\sigma_X = 2$ while Y has standard deviation $\sigma_Y = 4$.

(a) For $V = X - Y$, what are the smallest and largest possible values of $\mathrm{Var}[V]$?

(b) For $W = X - 2Y$, what are the smallest and largest possible values of $\mathrm{Var}[W]$?

a) $\mathrm{Var}(V) = \mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y)$

$\mathrm{Var}(X) = 4$, $\mathrm{Var}(Y) = 16$

$-1 \le \rho_{XY} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \le 1$

$\sigma_X = 2$, $\sigma_Y = 4$

$-8 \le \mathrm{Cov}(X,Y) \le 8$

$\mathrm{Var}(V) = 20 - 2\,\mathrm{Cov}(X,Y)$

$4 \le \mathrm{Var}(V) \le 36$

b) $W = X - 2Y$

$\mathrm{Var}(W) = \mathrm{Var}(X-2Y)$

$= E[(X-2Y)^2] - (E[X-2Y])^2$

$= E[X^2 - 4XY + 4Y^2] - (E[X])^2 + 4E[X]E[Y] - 4E[Y]^2$

$= E[X^2] - E[X]^2 + 4(E[Y^2] - E[Y]^2) - 4\,\mathrm{Cov}(X,Y)$

$= \mathrm{Var}(X) + 4\,\mathrm{Var}(Y) - 4\,\mathrm{Cov}(X,Y) = \mathrm{Var}(W)$

$\mathrm{Var}(X) = 4$, $\mathrm{Var}(Y) = 16$

$= 68 - 4\,\mathrm{Cov}(X,Y) = \mathrm{Var}(W)$

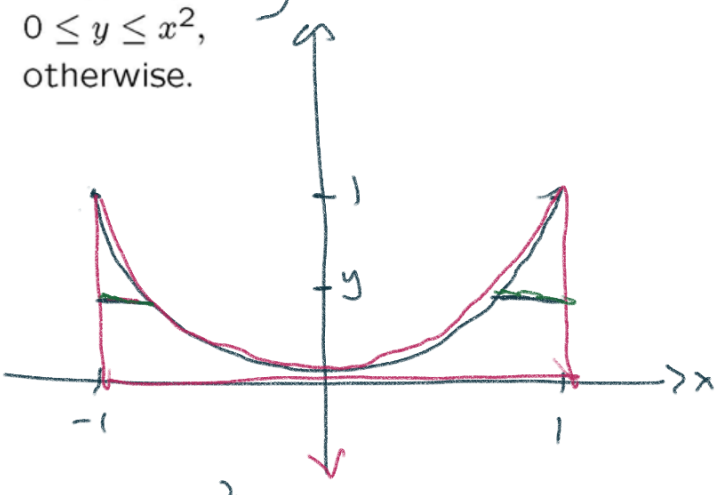
$= 68 - 4\,\rho_{XY}\,8 = 68 - 32\rho_{XY} = \mathrm{Var}(W)$

$$36 \le 68 - 32\rho_{XY} \le 100$$

Problem 5.7.13

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1; \\ & 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$



Answer the following questions.

(a) What are E[X] and Var[X]?
(b) What are E[Y] and Var[Y]?
(c) What is Cov[X,Y]?
(d) What is E[$X+Y$]?
(e) What is Var[$X+Y$]?

a) önce $f_X(x)$'i bulalım:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{x^2} \frac{5x^2}{2}\,dy$$

$$= \frac{5x^2 y}{2}\Big|_{y=0}^{x^2} = \frac{5x^4}{2}, \quad -1 \le x \le 1$$

$$E[X] = \int_{-1}^{1} x\,\frac{5x^4}{2}\,dx = \frac{5x^6}{12}\Big|_{-1}^{1} = 0$$ c)

$$\mathrm{Var}(X) = E[X^2] = \int_{-1}^{1} x^2\,\frac{5x^4}{2}\,dx = \frac{5x^7}{14}\Big|_{-1}^{1} = \frac{5}{7}$$

b) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x)\,dx$

$= \int_{-1}^{-\sqrt{y}} f_{XY}(x)\,dx + \int_{\sqrt{y}}^{1} f_{XY}(x)\,dx$

$= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\,dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\,dx$

$= \frac{5x^3}{6}\Big|_{-1}^{-\sqrt{y}} + \frac{5x^3}{6}\Big|_{\sqrt{y}}^{1} = \left(\frac{5}{6} - \frac{5}{6}y\sqrt{y}\right)2$

$0 \le y \le 1$

$E[Y] = \int_0^1 \frac{5y}{3} - \frac{5}{3}y^{3/2}\,y\,dy$

$= \frac{5y^2}{6} - \frac{5}{3}\frac{y^{7/2}}{7/2}\Big|_0^1 = \frac{5}{6} - \frac{5}{3}\frac{2}{7}$

$= \frac{5}{6} - \frac{10}{21} = \frac{35-20}{42} = \frac{15}{42}$

$E[Y^2] = \int_0^1 \frac{10y^2}{3} - \frac{10}{3}y^{7/2}\,dy = \frac{10y^3}{9} - \frac{10}{3}\frac{y^{9/2}}{9/2}\Big|_0^1$

$= \frac{10}{9} - \frac{10}{3}\times\frac{2}{9} = \frac{10}{9} - \frac{20}{27}$

$= \frac{10}{27}$

$\mathrm{Var}(Y) = \frac{10}{27} - \left(\frac{5}{14}\right)^2$

$= 0.24$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

($E(X) \to 0$, $E(Y) \to 5/14$)

$$E[XY] = \int_{x=-1}^{1}\int_{y=0}^{x^2} xy\,\frac{5x^2}{2}\,dy\,dx = \int_{x=-1}^{1}\int_{y=0}^{x^2} \frac{5x^3 y}{2}\,dy\,dx$$

$$\int_{x=-1}^{1}\left(\frac{5}{4}x^3y^2\Big|_{y=0}^{x^2}\right)dx = \int_{x=-1}^{1}\frac{5}{4}x^7\,dx$$

$$= \frac{5}{4}\frac{x^8}{8}\Big|_{-1}^{1} = 0$$

d) $E[X+Y] = E[X] + E[Y] = 0 + \frac{5}{14} = \frac{5}{14}$

e) $\mathrm{Var}(X+Y) = \underbrace{\mathrm{Var}(X)}_{5/7} + \underbrace{\mathrm{Var}(Y)}_{0.24} + 2\underbrace{\mathrm{Cov}(X,Y)}_{0}$

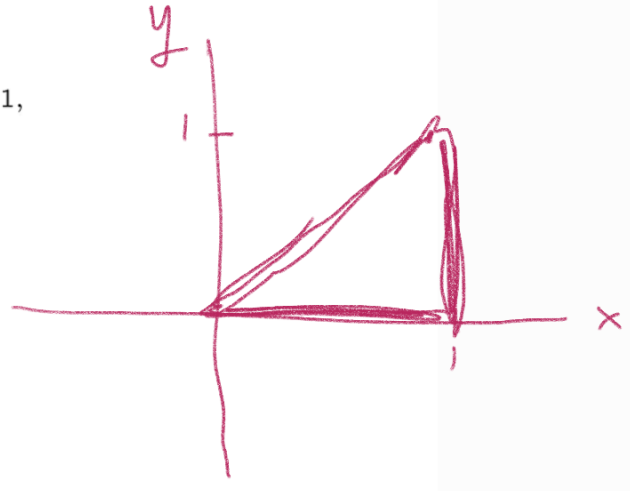
$$= 0.96$$

Problem 5.7.14

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What are $E[X]$ and $\mathrm{Var}[X]$?
(b) What are $E[Y]$ and $\mathrm{Var}[Y]$?
(c) What is $\mathrm{Cov}[X,Y]$?
(d) What is $E[X+Y]$?
(e) What is $\mathrm{Var}[X+Y]$?



Problem 5.7.15

A transmitter sends a signal X and a receiver makes the observation $Y = X + Z$, where Z is a receiver noise that is independent of X and $\mathrm{E}[X] = \mathrm{E}[Z] = 0$. Since the average power of the signal is $\mathrm{E}[X^2]$ and the average power of the noise is $\mathrm{E}[Z^2]$, a quality measure for the received signal is the signal-to-noise ratio

$$\Gamma = \frac{\mathrm{E}[X^2]}{\mathrm{E}[Z^2]}.$$

= sinyal gücü / gürültü gücü

How is Γ related to the correlation coefficient $\rho_{X,Y}$?

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{E}[X^2]}\,\sqrt{\mathrm{Var}(X+Z)}}$$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X, X+Z)}{\sqrt{\mathrm{Var}(X)}\,\sqrt{\mathrm{Var}(X)+\mathrm{Var}(Z)}}$$

X, Z bağımsız $\Rightarrow$ ilintisiz.

$$\mathrm{Cov}(X, X+Z) = \mathrm{E}[X^2 + XZ] - \mathrm{E}[X]\,\mathrm{E}[X+Z]$$

$$= \mathrm{E}[X^2] - \underline{\mathrm{E}[XZ]} - \mathrm{E}[X]\left(\mathrm{E}[X] + \underline{\mathrm{E}[Z]}\right)$$

$$\mathrm{Cov}(X, X+Z) = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \mathrm{Var}(X) = \mathrm{E}[X^2]$$

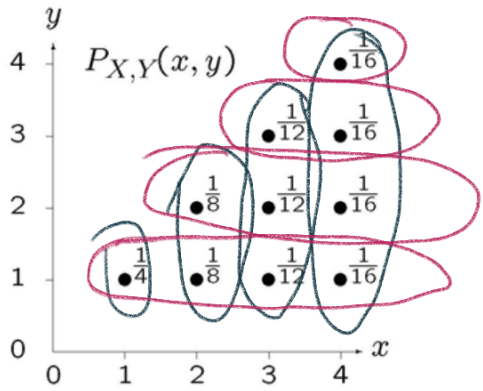
$$\rho_{X,Y} = \frac{\mathrm{Var}(X)}{\sqrt{\mathrm{Var}(X)}\,\sqrt{\mathrm{Var}(X)+\mathrm{Var}(Z)}} = \frac{\sqrt{\mathrm{Var}(X)}}{\sqrt{\mathrm{Var}(X)+\mathrm{Var}(Z)}}$$

$$= \frac{1}{\sqrt{1 + \dfrac{\mathrm{Var}(Z)}{\mathrm{Var}(X)}}} = \frac{1}{\sqrt{1 + \dfrac{\mathrm{E}[Z^2]}{\mathrm{E}[X^2]}}} = \frac{1}{\sqrt{1 + \dfrac{1}{\Gamma}}} = \rho_{X,Y}$$

Yani: $\Gamma \uparrow \iff \rho_{X,Y} \to 1$

Problem 5.8.6

The random variables X and Y have joint PMF



$P_X = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]$ $P_Y = \left[\frac{25}{48}, \frac{13}{48}, \frac{7}{48}, \frac{3}{48}\right]$

Find

(a) The expected values E[X] and E[Y],
(b) The variances Var[X] and Var[Y],
(c) The correlation, $r_{X,Y} = \mathrm{E}[XY]$,
(d) The covariance, Cov[X, Y],
(e) The correlation coefficient, $\rho_{X,Y}$.

$S_X = \{1,2,3,4\}$

$S_Y = \{1,2,3,4\}$

a) $E[X] = \frac{10}{4} = \frac{5}{2}$ $E[Y] = \frac{25+26+21+12}{48} = \frac{84}{48} = \frac{14}{8} = 7/4$

b) $E[X^2] = \frac{1+4+9+16}{4} = \frac{15}{2}$ $E[Y^2] = \frac{25+52+63+48}{48}$

$= \frac{188}{48} = \frac{47}{12}$

$Var(X) = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$ $Var(Y) = \frac{47}{12} - \frac{49}{16} = \frac{4.47 - 3.49}{48} = \frac{41}{48}$

c) $r = E[XY]$

$= 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 4 \times \frac{1}{8} + \frac{3}{12} + \frac{6}{12} + \frac{9}{12} + \frac{4}{16} + \frac{8}{16} + \frac{12}{16} + \frac{16}{16}$

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{18}{12} + \frac{40}{16}$

$= 1 + \frac{72}{48} + \frac{120}{48} = 5$

d) $\mathrm{Cov}(X,Y) = \underbrace{E(XY)}_{r} - \underbrace{E(X)}_{5/2}\underbrace{E(Y)}_{7/4}$

$= 5 - \frac{35}{8} = \frac{5}{8} > 0$

e) $\rho_{XY} = \frac{\mathrm{Cov}(XY)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{5/8}{\sqrt{\frac{5}{4}\cdot\frac{41}{48}}} = \frac{5}{8}\sqrt{\frac{192}{205}}$

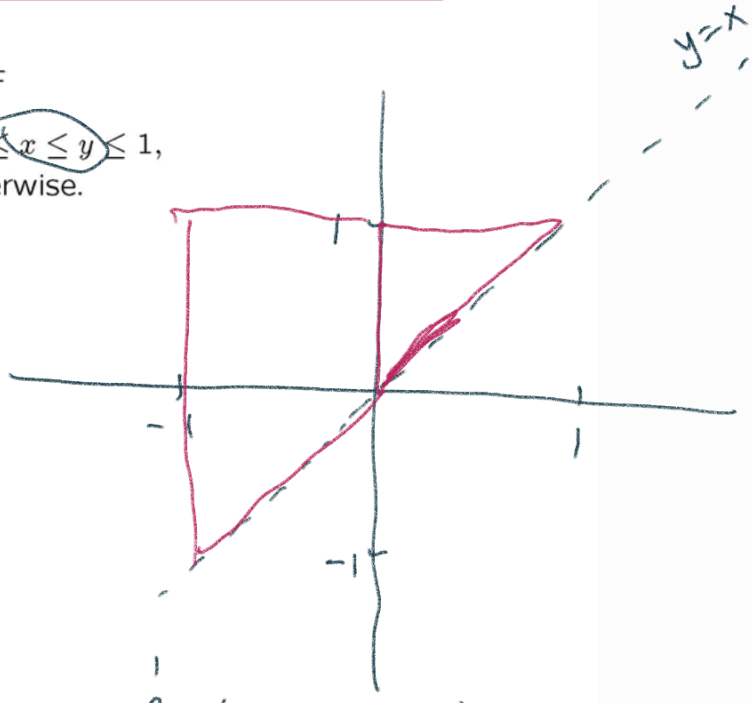
$= 0.6$

Problem 5.8.8

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $r_{X,Y}$ and $\mathrm{E}[e^{X+Y}]$.



$E(XY)$

$$E[XY] = \int_{y=-1}^{1}\int_{x=-1}^{y} xy\,\frac{1}{2}\,dx\,dy$$

$$\int_{y=-1}^{1}\left(\frac{x^2y}{4}\Big|_{x=-1}^{y}\right)dy = \int_{y=-1}^{1}\left(\frac{y^3}{4} - \frac{y}{4}\right)dy$$

$$= \frac{y^4}{16} - \frac{y^2}{8}\Big|_{y=-1}^{1} = \left(\frac{1}{16} - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{8}\right) = 0$$

$$E\left[e^{X+Y}\right] = \int_{y=-1}^{1}\int_{x=-1}^{y} e^x e^y \frac{1}{2}\,dx\,dy$$

$$= \int_{y=-1}^{1}\left(\frac{e^x e^y}{2}\Big|_{x=-1}^{y}\right)dy$$

$$= \int_{y=-1}^{1} \frac{e^{2y} - e^{y} e^{-1}}{2}\, dy = \left. \frac{e^{2y}}{4} - \frac{e^{y} e^{-1}}{2} \right|_{y=-1}^{1}$$

$$= \frac{e^{2}}{4} - \frac{e^{-2}}{4} - \frac{e^{0}}{2} + \frac{e^{-2}}{2}$$

$$E[e^{X+Y}] = \frac{e^{2}}{4} - \frac{1}{2} + \frac{e^{-2}}{4}$$

Problem 5.9.4

X_1 and X_2 are identically distributed Gaussian $(0,1)$ random variables. Moreover, they are jointly Gaussian. Under what conditions are X_1, X_2 and $X_1 + X_2$ identically distributed?

$$f_{XY}(x,y) = \frac{\exp\left[-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{XY}^2)} \right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}$$

$\rho_{XY} = 0$ (yani ilintisiz) oldugunda

$$f_{XY}(x,y) = \frac{\exp\left[-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2} \right]}{2\pi\sigma_X\sigma_Y}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2} \right) \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(\frac{-(y-\mu_Y)^2}{2\sigma_Y^2} \right)$$

$f_X(x)$ $f_Y(y)$

yani X, Y bagimsizdir.

$\mu_{X_1}, \mu_{X_2} = 0 \quad , \sigma_{X_1} = \sigma_{X_2} = 1$

$$f_{X_1 X_2}(x_1, x_2) = \exp\left[- \frac{x_1^2 - 2\rho_{X_1 X_2} x_1 x_2 + y^2}{2(1-\rho_{X_1 X_2}^2)} \right] \Bigg/ 2\pi\sqrt{1-\rho_{X_1 X_2}^2}$$

$X_1 \sim N(0,1)$

$X_2 \sim N(0,1)$

$X_1 + X_2 \sim N(0, 2+2\rho_{X_1 X_2})$

$Var(X_1+X_2) = \underbrace{Var(X_1)}_{1} + \underbrace{Var(X_2)}_{1} + 2\underbrace{Cov(X_1, X_2)}_{\rho_{X_1 X_2}\sigma_{X_1}\sigma_{X_2}}$

$= 2 + 2\rho_{X_1 X_2}$

Eğer $2 + 2\rho_{X_1 X_2} = 1$ olursa zaman X_1+X_2 ve X_1, X_2 aynı doğrultuda

Yani $\rho_{X_1 X_2} = \frac{-1}{2}$ olursa

Problem 5.8.10

Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1}p/n & k = 1,\ldots,n; \\ & n = 1,2,\ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PMF $P_N(n)$ and the expected values $\mathrm{E}[N]$, $\mathrm{Var}[N]$, $\mathrm{E}[N^2]$, $\mathrm{E}[K]$, $\mathrm{Var}[K]$, $\mathrm{E}[N+K]$, $r_{N,K}$, $\mathrm{Cov}[N,K]$.

$$P_N(n) = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}}{n} p = p \sum_{n=1}^{\infty} \frac{A^{n-1}}{n}$$

$$\sum_{n=0}^{\infty} (1-p)^n = \frac{1}{1-(1-p)} = \frac{1}{1-A}$$

$$\downarrow \frac{d}{dA}$$

$$\sum_{n=1}^{\infty} n(A)^{n-1}$$

Problem 5.9.5

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = ce^{-(2x^2-4xy+4y^2)}.$$

(a) What are $\mathrm{E}[X]$ and $\mathrm{E}[Y]$? → μ_X ve μ_Y ?

(b) Find the correlation coefficient $\rho_{X,Y}$.

(c) What are $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$?

(d) What is the constant c?

(e) Are X and Y independent?

$$f_{XY}(x,y) = \exp\left[-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{XY}^2)}\right] \Big/ 2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}$$

$$\frac{x^2}{\sigma_x^2} - 2x\frac{\mu_x}{\sigma_x} + \frac{\mu_x^2}{\sigma_x^2} - \frac{2\rho_{xy}\,xy}{\sigma_x\sigma_y} + \frac{2\rho_{xy}\,x\mu_y}{\sigma_x\sigma_y} + \frac{2\rho_{xy}\,y\mu_x}{\sigma_x\sigma_y} - \frac{2\rho_{xy}}{\sigma_x\sigma_y}\mu_x\mu_y + \frac{y^2}{\sigma_y^2} - \frac{2y\mu_y}{\sigma_y} + \frac{\mu_y^2}{\sigma_y^2}$$

a)

$$= 2(1-\rho_{xy}^2)\left(2x^2 - 4xy + 4y^2\right)$$

$$\frac{\mu_x^2}{\sigma_x^2} - \frac{2\rho_{xy}}{\sigma_x\sigma_y}\mu_x\mu_y + \frac{\mu_y^2}{\sigma_y^2} = 0$$

$$2\mu_x^2 - 4\mu_x\mu_y + 4\mu_y^2 = 0$$

$$\mu_x^2 - 2\mu_x\mu_y + 2\mu_y^2 = 0$$

$$(\mu_x - \mu_y)^2 + \mu_y^2 = 0 \Rightarrow \mu_x = 0,\ \mu_y = 0$$

$$-\frac{2\mu_x}{\sigma_x} + \frac{2\rho_{xy}\mu_y}{\sigma_x\sigma_y} = 0$$

c)

$$\frac{x^2}{\sigma_x^2} = 2x^2$$

$$\frac{y^2}{\sigma_y^2} = 4y^2$$

$$\sigma_x^2 = \frac{1}{2}$$

$$\sigma_y^2 = \frac{1}{4}$$

$$\mu_x = \rho_{xy}\frac{\mu_y}{\sigma_y} \Rightarrow \mu_x = 2\sigma_x\mu_y$$

$$\frac{2\rho_{xy}\mu_x}{\sigma_x\sigma_y} = \frac{2\mu_y}{\sigma_y}$$

$$\mu_x = \frac{\mu_y\sigma_x}{\rho_{xy}} \Rightarrow \mu_x = \frac{\mu_y}{2\sigma_y}$$

b)

$$-\frac{2\rho_{xy}}{\sigma_x\sigma_y}xy = -4xy$$

$$\Downarrow$$

$$\rho_{xy} = 2\sigma_x\sigma_y$$

$$\Downarrow$$

$$\rho_{xy} = 2 \cdot \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$

$$\rho_{xy} = \frac{1}{\sqrt{2}} \quad *$$

d) $c \Rightarrow$

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} = \frac{1}{2\pi\frac{1}{\sqrt{2}}\frac{1}{2}\sqrt{1-\frac{1}{2}}} = \frac{2}{\pi} \;//$$

e) $\rho_{XY} \neq 0 \Rightarrow$ Bağımsız değil

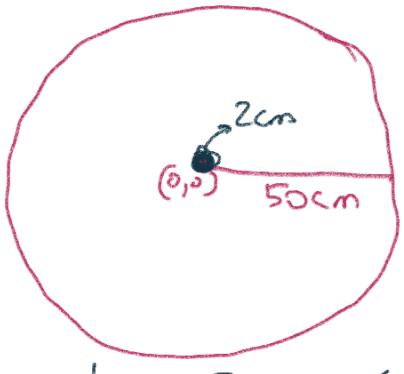
Problem 5.9.6

An archer shoots an arrow at a circular target of radius 50 cm. The arrow pierces the target at a random position (X, Y), measured in centimeters from the center of the disk at position $(X, Y) = (0, 0)$. The bullseye is a solid black circle of radius 2 cm, at the center of the target. Calculate the probability $P[B]$ of the event that the archer hits the bullseye under each of the following models:

(a) X and Y are iid continuous uniform $(-50, 50)$ random variables.

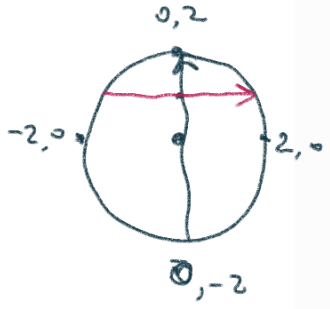
(b) The PDF $f_{X,Y}(x, y)$ is uniform over the 50 cm circular target.

(c) X and Y are iid Gaussian $(\mu = 0, \sigma = 10)$ random variables.



$$f_X(x) = \frac{1}{\sqrt{2\pi}\,10} e^{-x^2/200}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,10} e^{-y^2/200}$$



a) $f_X(x) = \frac{1}{100}, -50 \leq x \leq 50$

$f_Y(y) = \frac{1}{100}, -50 \leq y \leq 50$

Hedefi vurmak $\Rightarrow$ $X^2 + Y^2 \leq 4$

$$P(B) = P(X^2 + Y^2 \leq 4)$$

$$\int_{y=-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{100} \frac{1}{100} \, dx \, dy$$

$$= \int_{-2}^{2} \frac{2\sqrt{4-y^2}}{10000} \, dy = \int_{-2}^{2} \frac{4\sqrt{1-\left(\frac{y}{2}\right)^2}}{10000} \, dy$$

$\sin\theta = \frac{y}{2}$

$\cos\theta \, d\theta = dy/2$

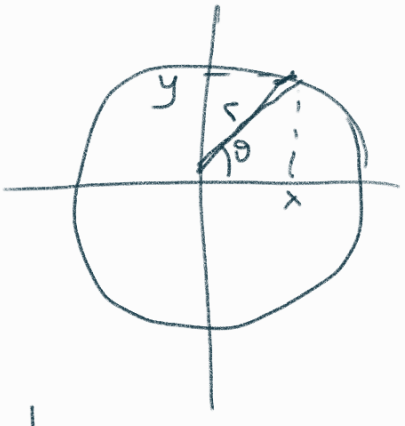
$$P(B) = \frac{4}{1000}\int_{-\pi/2}^{\pi/2} \cos\theta \times 2\cos\theta \, d\theta$$

$$= \frac{4}{10000}\int_{-\pi/2}^{\pi/2} \cos(2\theta)+1 \quad d\theta \qquad 2\cos^2\theta = \cos(2\theta)+1$$

$$= \frac{4}{1000}\left.\frac{\sin(2\theta)}{2} + \theta\right|_{-\pi/2}^{\pi/2} = \frac{4\pi}{10000} \checkmark$$

b) $P(B) = \dfrac{\text{Siyah hedef alanı}}{\text{Daire alanı}} = \dfrac{\pi 2^2}{\pi 50^2} = \dfrac{4}{2500}$

c) $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ $\quad x^2+y^2 = r^2$



$$\Rightarrow \iint_{x^2+y^2\le 4} \frac{1}{2\pi 100} e^{-\frac{x^2+y^2}{200}} \, dx\,dy$$

$$\int_0^{2\pi}\int_0^2 \frac{1}{200} e^{-r^2/200} \, r \, dr\,d\theta$$

Problem 5.10.7

X_1, X_2, X_3 are iid exponential (λ) random variables. Find:
(a) the PDF of $V = \min(X_1, X_2, X_3)$,
(b) the PDF of $W = \max(X_1, X_2, X_3)$.

ϑ

$2r\,dr = du$

$$\int_{\theta=0}^{2\pi} \left(\int_{u=0}^{4} \frac{1}{200\pi} e^{-u/200} \frac{du}{2} \right) d\theta$$

$$\int_{\theta=0}^{2\pi} \frac{1}{2\pi}\left(1-e^{-4/200}\right) d\theta = 1-e^{-4/200}$$

Problem 5.10.9

Random variables $X_1, X_2, \ldots, X_n$ are iid; each X_j has CDF $F_X(x)$ and PDF $f_X(x)$. Consider

$$L_n = \min(X_1, \ldots, X_n)$$
$$U_n = \max(X_1, \ldots, X_n).$$

In terms of $F_X(x)$ and/or $f_X(x)$:

(a) Find the CDF $F_{U_n}(u)$.

(b) Find the CDF $F_{L_n}(l)$.

(c) Find the joint CDF $F_{L_n,U_n}(l,u)$.

a) $F_{U_n}(u) = P(U_n \le u) = P(\max(X_1, \ldots, X_n) \le u)$

$= P(X_1 \le u, X_2 \le u, \ldots, X_n \le u)$ (independent)

$= \prod_{i=1}^{n} P(X_i \le u) = \prod_{i=1}^{n} F_X(u)$ (identical dist.)

$= (F_X(u))^n$

b) $F_{L_n}(l) = P(L_n \le l)$

$= P(\min(X_1, \ldots, X_n) \le l)$

$= 1 - P(\min(X_1, \ldots, X_n) > l)$

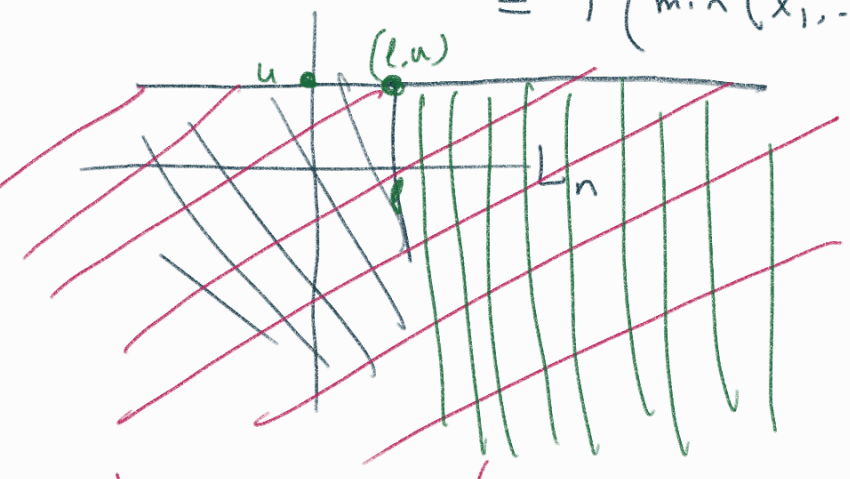
$= 1 - P(X_1 > l, X_2 > l, \ldots, X_n > l)$

$= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(l))$

$$= 1 - \left(1 - F_X(\ell)\right)^n$$

c) $F_{L_n U_n}(\ell, u) = P(L_n \le \ell \,,\, U_n \le u)$

U_n $= P(\min(X_1, \dots, X_n) \le \ell \,,\, \max(X_1, \dots, X_n) \le u)$



Kırmızı : $P(\max(X_1, \dots, X_n) \le u) = \left(F_X(u)\right)^n$

Yeşil : $P(\max(X_1, \dots, X_n) \le u, \min(X_1, \dots, X_n) \ge \ell)$

$= P(\ell \le X_1 \le u, \ell \le X_2 \le u, \dots, \ell \le X_n \le u)$

$= \underbrace{P(\ell \le X_1 \le u)}_{F_X(u) - F_X(\ell)} \underbrace{P(\ell \le X_2 \le u)}_{=} \cdots \underbrace{P(\ell \le X_n \le u)}_{=}$

$= \left(F_X(u) - F_X(\ell)\right)^n$

$F_{U_n L_n}(\ell, u) = \left(F_X(u)\right)^n - \left(F_X(u) - F_X(\ell)\right)^n$

Problem 5.10.6

The joint PMF of X, Y, and Z, the number of 1-page, 2-page, and 3-page downloads, respectively, in each of four downloads is given by

$$P_{X,Y,Z}(x,y,z) = \binom{4}{x,y,z}\frac{1}{3^x}\frac{1}{2^y}\frac{1}{6^z}.$$

multinomial

(a) In a group of four downloads, what is the PMF of the number of 3-page documents?
(b) In a group of four downloads, what is the expected number of 3-page documents?
(c) Given that there are two 3-page documents in a group of four, what is the joint PMF of the number of 1-page documents and the number of 2-page documents?
(d) Given that there are two 3-page documents in a group of four, what is the expected number of 1-page documents?
(e) In a group of four downloads, what is the joint PMF of the number of 1-page documents and the number of 2-page documents?

$\frac{1}{3}$: 1 page $\frac{1}{2}$: 2 page $\frac{1}{6}$: 3 page

a) Binom (3 page veya değil)

$p=\frac{1}{6}$ $n=4$

$$P_Z(z) = \binom{4}{z}\left(\frac{1}{6}\right)^z\left(\frac{5}{6}\right)^{4-z}$$

b) $E[Y] = 4\times\frac{1}{6} = \frac{2}{3}$

c) Kalan 2 download ve bunlar ya 2 ya da 1 page.

$P[1\text{ page}] = \frac{\frac{1}{3}}{\frac{5}{6}} = \frac{2}{5}$ (X) $\quad P(2\text{ page}) = \frac{\frac{1}{2}}{\frac{5}{6}} = \frac{3}{5}$ (Y)

$$P_{XY}(x,y) = \binom{2}{x,y}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^y \quad x,y\in\{0,1,2\},\ x+y\le 2$$

Binom

$$P[1\text{ page} \mid \text{not 3 page}] = \frac{P[1\text{ page}, \text{not 3 page}]}{P[\text{not 3 page}]} = \frac{P(1\text{ page})}{P[\text{not 3 page}]} = \frac{1/3}{5/6} = \frac{2}{5}$$

d) $E[X \mid 2\text{ 3-page downloads}]$

$= 2\times\frac{2}{5} = \frac{4}{5}$

e) 4 download $P[1\text{ page}]=\frac{1}{3}$ $P[2\text{ page}]=\frac{1}{2}$ $P[3\text{ page}]=\frac{1}{6}$

$$P_{XY}(x,y)=\binom{4}{x,y,4-x-y}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{4-x-y}$$

başta verilen olasılık ağırlık fonksiyonu ile aynı

Not: $\binom{2}{x,y}=\frac{2!}{x!\,y!}$

Problem 5.9.9

Your course grade depends on two test scores: X_1 and X_2. Your score X_i on test i is Gaussian ($\mu = 74, \sigma = 16$) random variable, independent of any other test score. $\sigma^2 = 256$

(a) With equal weighting, grades are determined by $Y = X_1/2 + X_2/2$. You earn an A if $Y \geq 90$. What is $\mathrm{P}[A] = \mathrm{P}[Y \geq 90]$?

(b) A student asks the professor to choose a weight factor w, $0 \leq w \leq 1$, such that

$$Y = wX_1 + (1-w)X_2.$$

Find $\mathrm{P}[A]$ as a function of the weight w. What value or values of w maximize $\mathrm{P}[A] = \mathrm{P}[Y \geq 90]$? $1/2 = w$

(c) A different student proposes that the better exam is the one that should count and that grades should be based on $M = \max(X_1, X_2)$. In a fit of generosity, the professor agrees! Now what is $\mathrm{P}[A] = \mathrm{P}[M > 90]$?

(d) How generous was the professor? In a class of 100 students, what is the expected increase in the number of A's awarded?

a) $Y \sim N(74, 8\sqrt{2})$

$$E[Y] = \frac{E[X]}{2} + \frac{E[X]}{2} = 74$$

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(X)}{4} + \frac{\mathrm{Var}(X)}{4} = \frac{256}{4} \times 2 = 128$$

$\sigma_Y = 8\sqrt{2}$

$$P(Y \geq 90) = P\left(\frac{Y-74}{8\sqrt{2}} \geq \frac{90-74}{8\sqrt{2}}\right) = Q\left(\frac{16}{8\sqrt{2}}\right) = Q(\sqrt{2})$$

$\frac{Y-74}{8\sqrt{2}} \sim N(0,1)$

$$Q(A) = \int_A^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

b) $Y = wX_1 + (1-w)X_2 \sim N\left(74, \sqrt{256w^2 + (1-w)^2 256}\right)$

$E[Y] = 74$

$Var(Y) = 256w^2 + (1-w)^2 256$

$= 256\left(w^2 + 1 - 2w + w^2\right)$

$= 256\left(2w^2 - 2w + 1\right)$

$$P(Y \geq 90) = P\left(\frac{Y-74}{8\sqrt{2}\sqrt{2w^2-2w+1}} \geq \frac{90-74}{8\sqrt{2}\sqrt{2w^2-2w+1}}\right)$$

$$= Q\left(\frac{\sqrt{2}}{\sqrt{2w^2-2w+1}}\right) \Rightarrow$$ bunun maksimize etmek istiyoruz

A'yı minimize etmeli

$$\min \sqrt{\frac{2}{2w^2-2w+1}} \Rightarrow \max\left\{2w^2-2w+1\right\}$$

$4w - 2 = 0$

$\boxed{w = 1/2}$

c) $M = \max(X_1, X_2)$

$$P(M \geq 90) = P(\max(X_1, X_2) \geq 90)$$

$$= 1 - P(\max(X_1, X_2) < 90)$$

$$= 1 - P(X_1 < 90)\,P(X_2 < 90)$$

$= 1 - \left(\Phi\left(\frac{90-74}{16}\right)\right)^2$

$= 1 - \left(1 - Q(1)\right)^2$

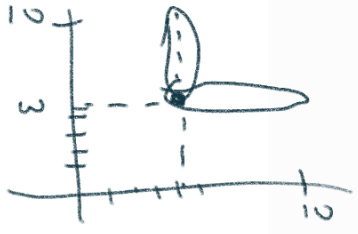
d) $Q(\sqrt{2}) \leq 1 - (1 - Q(1))^2$ ✓

$100\,Q(\sqrt{2}) \overset{?}{\leq} 100\left[1 - (1 - Q(1))^2\right]$

Problem 6.1.5

Let X and Y be discrete random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} 0.01 & x = 1,2\ldots,10, \\ & y = 1,2\ldots,10, \\ 0 & \text{otherwise.} \end{cases}$$



What is the PMF of $W = \min(X,Y)$?

Problem 6.1.6

For random variables X and Y in Problem 6.1.5, what is the PMF of $Y = \max(X,Y)$?

6.1.5

$P(W = w) = P(\min(X,Y) = w)$

$= \frac{11-w}{100} + \frac{11-w}{100} - \frac{1}{100}$

$P(W = w) = \frac{21-2w}{100}$

$P(W = 10) = \frac{1}{100} = P(X = 10, Y = 10)$

$P(W = 1) = \frac{19}{100}$

6.1.6

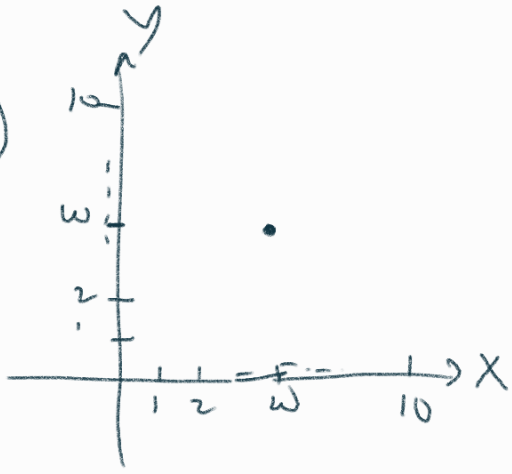
$W = \max(X, Y)$

$P(W=w) = P(\max(X,Y) = w)$

$= P(X=w, Y \le w) + P(X \le w, Y=w) - P(X=Y=w)$

$\frac{w}{100} + \frac{w}{100} - \frac{1}{100}$

$P(W=w) = \frac{2w-1}{100}$



Problem 6.2.6

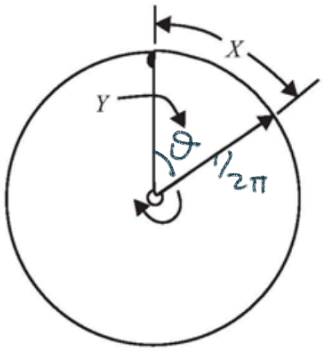
Let X denote the position of the pointer after a spin on a wheel of circumference 1. For that same spin, let Y denote the area within the arc defined by the stopping position of the pointer:

$2\pi r = 1$

$r = \frac{1}{2\pi}$

$A = \pi r^2 = \pi \frac{1}{4\pi^2}$

$A = \frac{1}{4\pi}$



$0 \le X \le 1$

düzgün.

$0 \le Y \le \frac{1}{4\pi}$

(a) What is the relationship between X and Y?
(b) What is $F_Y(y)$?
(c) What is $f_Y(y)$?
(d) What is E[Y]?

a) $X = 1 \times \frac{\theta}{2\pi}$ $\quad Y = \frac{1}{4\pi} \times \frac{\theta}{2\pi}$

$Y\,4\pi = X$

b) $F_Y(y) = P(Y \le y) = P\left(\frac{X}{4\pi} \le y\right) = P(X \le 4\pi y)$

$F_Y(y) = 4\pi y$

$0 \le y \le \frac{1}{4\pi}$

c) $f_Y(y) = \dfrac{dF_Y(y)}{dy} = 4\pi$, $0 \le y \le \frac{1}{4\pi}$

düzgün.

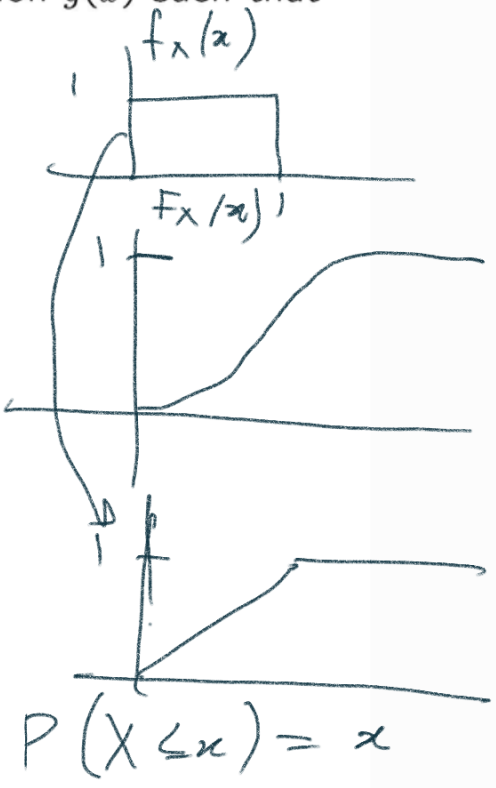
d) $E[Y] = \dfrac{1}{8\pi}$

Problem 6.2.8

X is the uniform $(0,1)$ random variable. Find a function $g(x)$ such that the PDF of $Y = g(X)$ is

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$F_Y(y) = \begin{cases} y^3 & 0 \le y \le 1 \\ 1 & 1 < y \end{cases}$$



$$P(Y \le y) = y^3 \qquad 0 \le y \le 1$$

$$P(g(X) \le y) = y^3 \qquad 0 \le y \le 1$$

$$P(X \le g^{-1}(y)) = y^3 \Rightarrow$$

$g(x)$ monoton ise

$$g^{-1}(y) = y^3$$

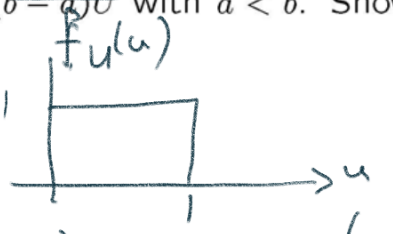
$$\boxed{g(x) = x^{1/3}}$$

$$\left[F_X(x) \quad U \sim \text{rand}(0,1) \Rightarrow F_X^{-1}(U) \sim f_X(x) \right]$$

rastgele sayı üretme

Problem 6.2.11

For the uniform $(0,1)$ random variable U, find the CDF and PDF of $Y = a + (b-a)U$ with $a < b$. Show that Y is the uniform (a,b) random variable.



$$P(Y \le y) = P(a + (b-a)U \le y)$$

$$= P\left(U \le \frac{y-a}{b-a}\right) = \frac{y-a}{b-a}, \quad 0 < \frac{y-a}{b-a} < 1$$

$$F_Y(y) = \begin{cases} \frac{y-a}{b-a}, & a \le y \le b \\ 0 & \text{o.w} \end{cases}$$

$\frac{d}{dy}$ ↓

$$f_Y(y) = \begin{cases} \frac{1}{b-a} & a \le y \le b \\ 0 & \text{o.w.} \end{cases}$$

[$Y = a + (b-a)U \longrightarrow$ a, b arası düzgün dağılımlı]

↓

0,1 arası düzgün dağılımlı

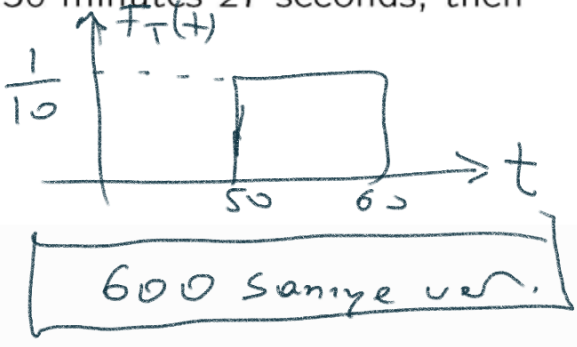
$-\ln(\text{rand}) \sim \exp(1)$

$-\frac{1}{\lambda}\log(\text{rand}) \sim \exp(\lambda)$

Problem 6.3.2

In a 50 km cycling time trial, a rider's exact time T, measured in minutes, is the continuous uniform (50,60) random variable. However, a rider's recorded time R in seconds is obtained by rounding *up* T to next whole second. That is, if T is 50 minutes, 27.001 seconds, then $R = 3028$ seconds. On the other hand, if T is exactly 50 minutes 27 seconds, then $R = 3027$. What is the PMF of R?

$T = U(50, 60)$ → dk



R → sn

50 dk 27,001 sn ⟹ 50 dk , 28 sn

3028 sn

$$P(R=r) = P\left(\left\lfloor \frac{r}{60} \right\rfloor < T \leq \left\lceil \frac{r}{60} \right\rceil\right), \quad 3000 \leq r \leq 3600$$

3281 ⟹ $P(54 < T \leq 55) = \frac{1}{600}$

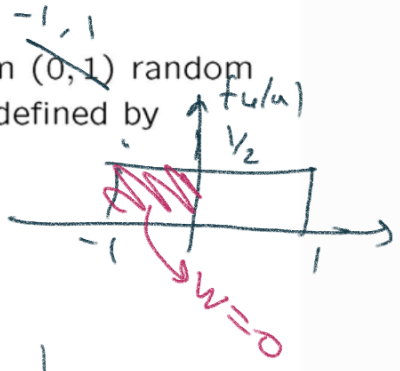
3281 | 60
300 | 54
281
240
41

Problem 6.3.7

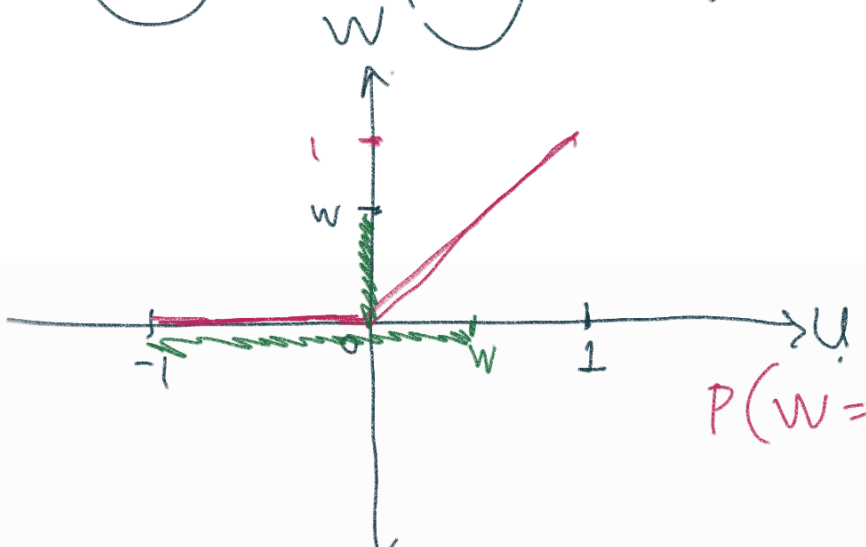
The input voltage to a rectifier is the continuous uniform (0, 1) random variable U. The rectifier output is a random variable W defined by

$$W = g(U) = \begin{cases} 0 & U < 0, \\ U & U \geq 0. \end{cases}$$

Find the CDF $F_W(w)$ and the expected value E[W].



$S_W = [0, 1]$

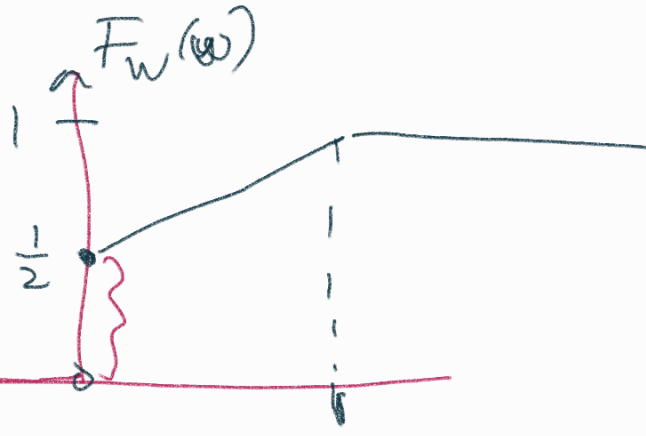


$$P(W=0) = \underbrace{P(U<0)}_{\frac{1}{2}}$$

$$P(W \leq w) = P(U \leq w), \quad 0 < w < 1$$

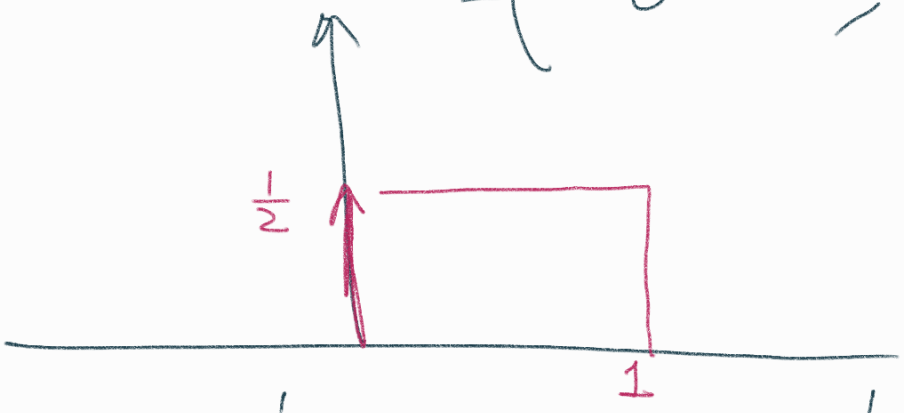
$$P(W \leq w) = 0, \quad w < 0$$

$$\boxed{P(W=0) = \frac{1}{2}}$$



$$F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{1}{2} + \frac{w}{2} & 0 \leq w < 1 \\ 1 & 1 \leq w \end{cases}$$

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} \frac{1}{2}\delta(w) + \frac{1}{2} & , 0 \le w \le 1 \\ 0 & , \text{o.w} \end{cases}$$



$$E[W] = \int_0^1 w\, f_W(w)\, dw = \int_0^1 w \left(\frac{1}{2}\delta(w) + \frac{1}{2}\right) dw$$

$$= \underbrace{\int_0^1 \frac{w}{2}\delta(w)\, dw}_{0} + \underbrace{\left.\frac{w^2}{4}\right|_0^1}_{\frac{1}{4}}$$

$$E[W] = \frac{1}{4}$$

Problem 6.3.10

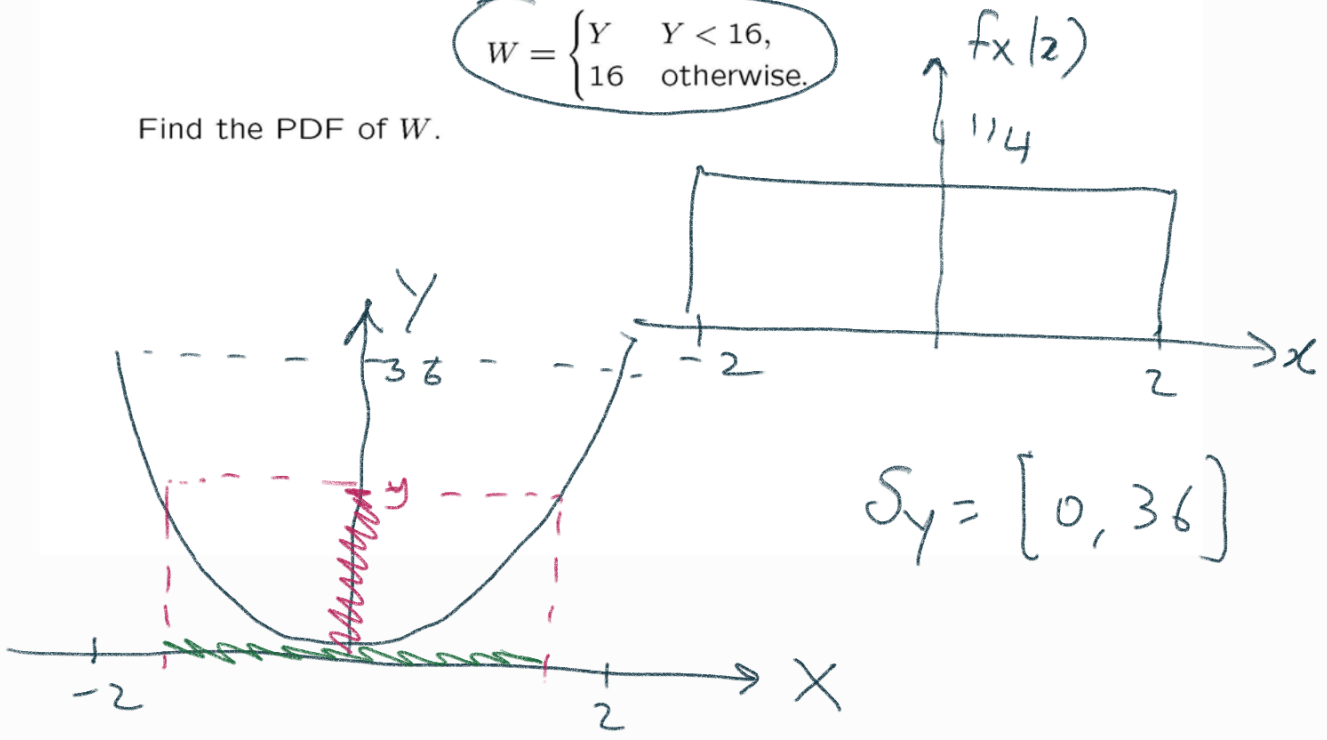
The current X across a resistor is the continuous uniform $(-2, 2)$ random variable. The power dissipated in the resistor is $Y = 9X^2$ Watts.

(a) Find the CDF and PDF of Y.

(b) A power measurement circuit is range-limited so that its output is

$$W = \begin{cases} Y & Y < 16, \\ 16 & \text{otherwise.} \end{cases}$$

Find the PDF of W.

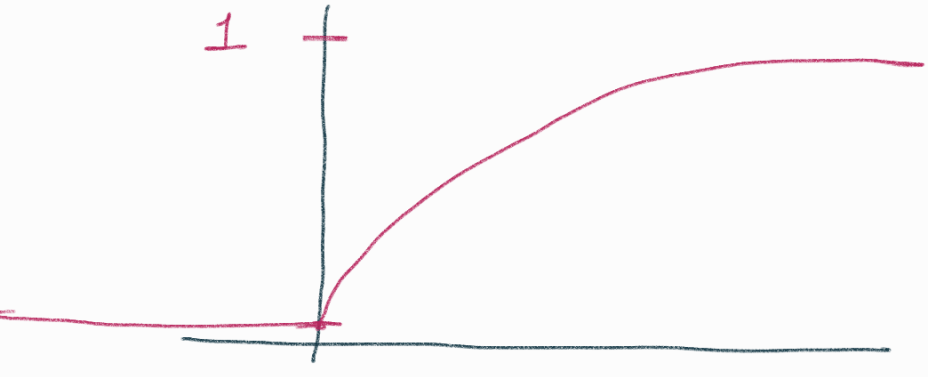


$S_Y = [0, 36]$

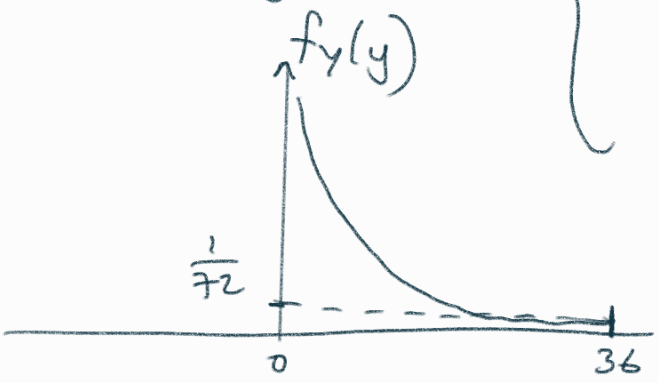
$$P(Y \le y) = P(9X^2 \le y) = P\left(\frac{-\sqrt{y}}{3} \le X \le \frac{\sqrt{y}}{3}\right)$$

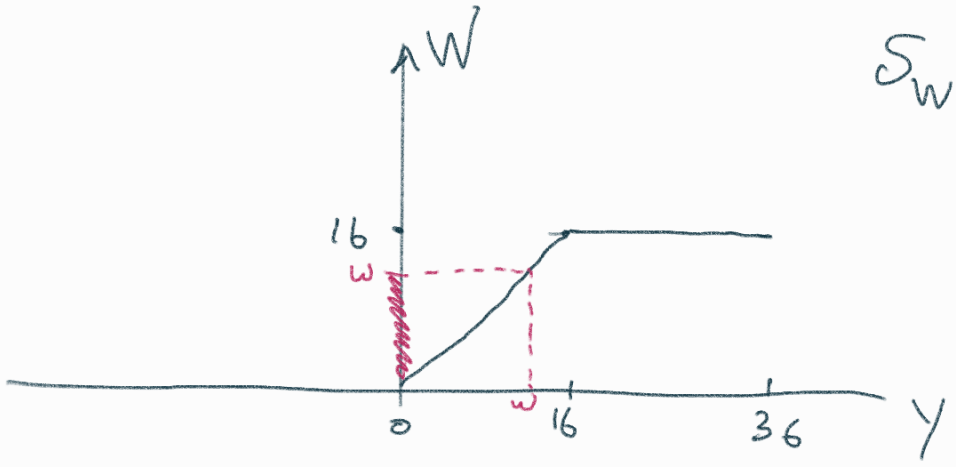
$$= \int_{-\frac{\sqrt{y}}{3}}^{\frac{\sqrt{y}}{3}} \frac{1}{4}\, dy = \frac{2\sqrt{y}}{12} = \frac{\sqrt{y}}{6}, \quad 0 \le y \le 36$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{\sqrt{y}}{6} & 0 \le y \le 36 \\ 1 & 36 < y \end{cases}$$



$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{12\sqrt{y}} & , 0 \le y \le 36 \\ 0 & \text{o.w.} \end{cases}$$

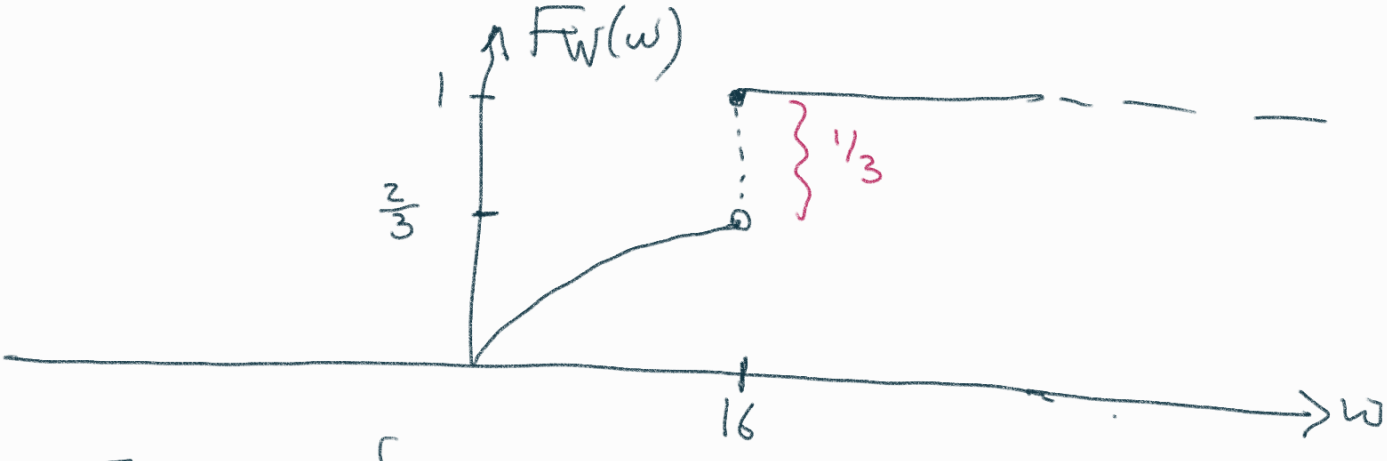




$S_W = [0, 16]$
mixed.

$$P(W \le w) = P(Y \le w) \quad , \quad w < 16$$
$$= \frac{\sqrt{w}}{6} \quad , 0 \le w < 16$$

$$P(W \le w) = 1 \quad 16 \le w .$$



$$F_W(w) = \begin{cases} 0 & w \le 0 \\ \sqrt{w}/6 & 0 < w < 16 \\ 1 & w \ge 16 \end{cases}$$

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} \frac{1}{12\sqrt{w}} + \frac{1}{3}\delta(w-16) & 16 \geq w > 0 \\ 0 & w \leq 0 \end{cases}$$

$$E[W] = \int_0^{16} \frac{w}{12\sqrt{w}} + \frac{w}{3}\delta(w-16)\, dw$$

$$= \int_0^{16} \frac{\sqrt{w}}{12}\, dw + \int_0^{16} \frac{w}{3}\delta(w-16)\, dw$$

$$\frac{1}{18} w^{3/2} \Big|_0^{16} + \frac{16}{3} = \frac{64}{18} + \frac{16}{3} = \frac{32}{9} + \frac{16}{3} = \frac{80}{9}$$

HATIRLA

$$\int_{-\infty}^{\infty} g(x)\delta(x-x_0)\, dx = \int_{-\infty}^{\infty} g(x_0)\delta(x-x_0)\, dx$$

$$= g(x_0) \underbrace{\int_{-\infty}^{\infty} \delta(x-x_0)\, dx}_{1}$$

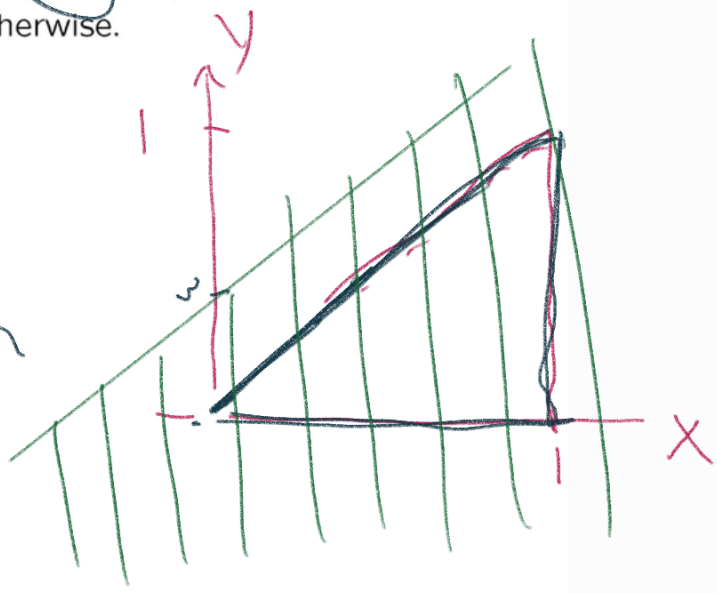
$$= g(x_0)$$

Problem 6.4.5

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6y & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = Y - X$.
(a) What is S_W, the range of W?
(b) Find $F_W(w)$ and $f_W(w)$.



$W = Y - X$

a) S_W range, değer aralığı

$W = Y - X$

$Y \le X \Rightarrow Y - X \le 0$

$X = 1$ olabilir $Y = 0$ olabilir. $\Rightarrow Y - X = -1$

$S_X = [-1, 0] \Rightarrow -1 \le W \le 0$

b) $P(W \le w) = P(Y - X \le w) = P(Y \le X + w)$

$Y = X + w$

$w > 0$ için $P(Y - X \le w) = 1$

$-1 < w < 0$ için

$$P(Y - X \le w) = \int_{x=-w}^{1} \int_{y=0}^{x+w} 6y \, dy \, dx$$

$$\int_{x=-w}^{1} \left(3y^2 \Big|_{0}^{x+w} \right) dx = \int_{x=-w}^{1} 3x^2 + 6xw + 3w^2 \, dx$$

$$= x^3 + 3x^2 w + 3w^2 x \Big|_{x=-w}^{1}$$

$$= \left(1 + 3w + 3w^2\right) - \left(-w^3 + 3w^3 - 3w^3\right)$$

$$P(W \le w) = P(Y - X \le w) = w^3 + 3w^2 + 3w + 1 \quad , -1 < w < 0$$

$$w < -1 \Rightarrow P(W \le w) = 0$$

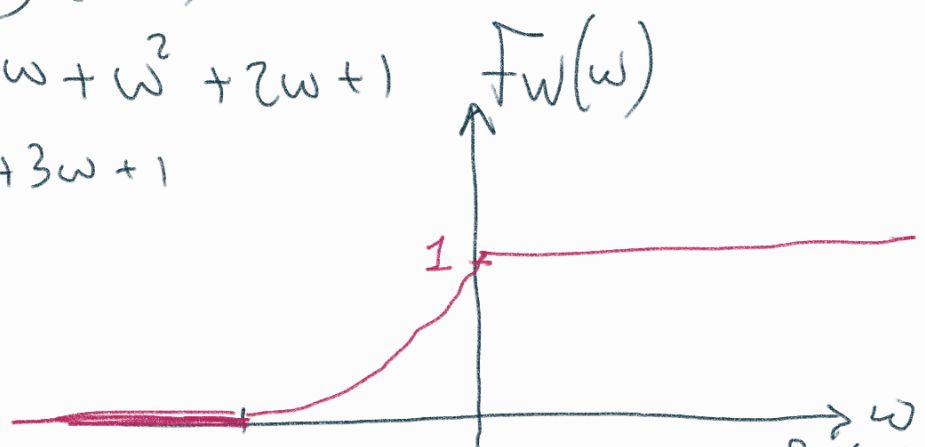
$$F_W(w) = \begin{cases} 1 & w > 0 \\ w^3 + 3w^2 + 3w + 1 & -1 \le w \le 0 \\ 0 & w \le -1 \end{cases}$$

$$w^3 + 3w^2 + 3w + 1 \rightarrow (w+1)^3$$

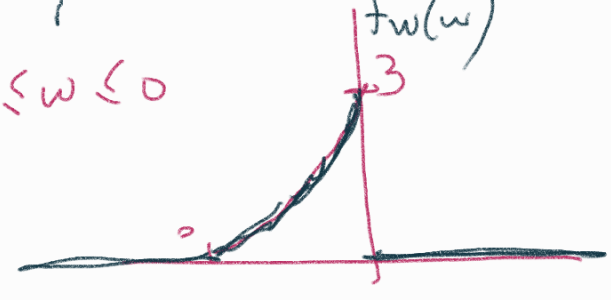
$$(w+1)^3 = (w^2 + 2w + 1)(w+1)$$

$$= w^3 + 2w^2 + w + w^2 + 2w + 1$$

$$= w^3 + 3w^2 + 3w + 1$$

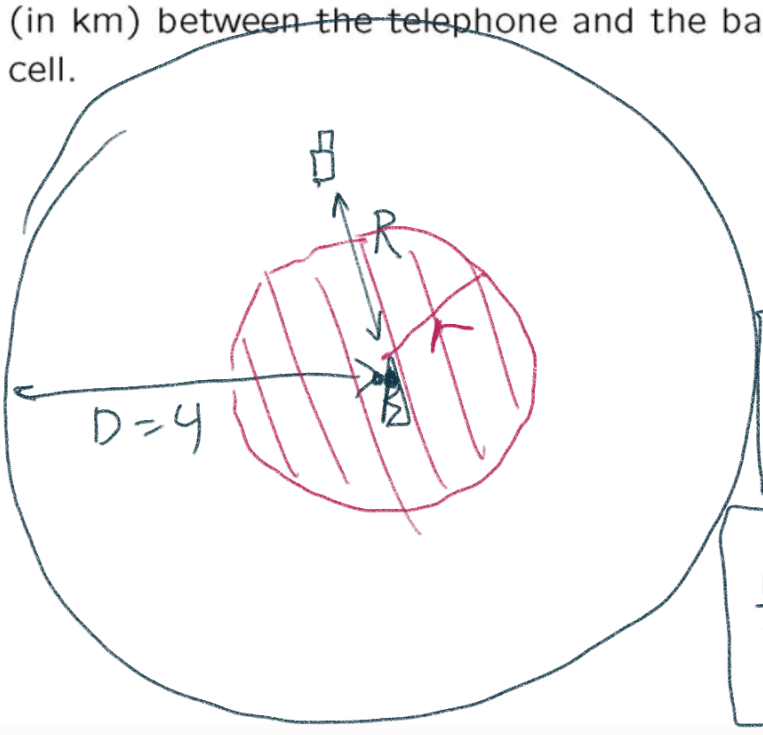


$$f_W(w) = \frac{d}{dw} F_W(w) = 3(w+1)^2 \quad , -1 \le w \le 0$$



Problem 6.4.8

In a simple model of a cellular telephone system, a portable telephone is equally likely to be found anywhere in a circular cell of radius 4 km. (See Problem 5.5.4.) Find the CDF $F_R(r)$ and PDF $f_R(r)$ of R, the distance (in km) between the telephone and the base station at the center of the cell.



$$F_R(r) = P(R < r)$$

$$= \frac{\pi r^2}{\pi D^2}$$

$$F_R(r) = r^2/D^2$$

$$f_R(r) = \frac{2r}{D^2}$$

$$\frac{r^2}{D^2} = U \Rightarrow r = \sqrt{U D^2} = D\sqrt{U} = D\sqrt{\text{rand}}$$

U ↳ uniform $(0,1)$ "rand"

$\theta = 2\pi * \text{rand}$

Problem 6.4.9

X and Y are independent identically distributed Gaussian $(0,1)$ random variables. Find the CDF of $W = X^2 + Y^2$.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \qquad -\infty < x, y < \infty$$

$$F_W(w) = P(W \le w) = \iint_{x^2+y^2 \le w} \frac{1}{2\pi} e^{-(x^2+y^2)/2} \, dy \, dx$$

$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$) Kutupsal koordinat sistemine geçiş

$$\int_{\theta=0}^{2\pi} \int_{r=-\sqrt{w}}^{\sqrt{w}} \frac{1}{2\pi} e^{-r^2/2} \, r\,dr\,d\theta = \int_{r=-\sqrt{w}}^{\sqrt{w}} \left(\frac{1}{2\pi} e^{-r^2/2} \, r\theta \Big|_0^{2\pi} \right) dr$$

$$= \int_{r=-\sqrt{w}}^{\sqrt{w}} e^{-r^2/2} \, r \, dr$$

$r^2 = u \qquad 2r\,dr = du \Rightarrow$

$$P(X^2+Y^2 \le w) = \int_{u=0}^{w} e^{-u/2} \frac{1}{2} du = -e^{-u/2} \Big|_0^w$$

$$F_W(w) = \boxed{P(W \le w) = 1 - e^{-w/2}}$$

$$f_W(w) = \frac{1}{2} e^{-w/2}, \quad w > 0 \Rightarrow \text{exponential}\left(\frac{1}{2}\right) \text{ üstel}$$

(Not: $\left(\frac{X}{\sqrt{2}}\right)^2 + \left(\frac{Y}{\sqrt{2}}\right)^2 = \frac{X^2+Y^2}{2} \sim$ üstel (1))

Problem 6.4.11

X is the Gaussian $(0,1)$ random variable and Z, independent of X, has PMF

$$P_Z(z) = \begin{cases} 1-p & z=-1, \\ p & z=1. \end{cases}$$

$Z=-1 \Rightarrow Y=-X \sim N(0,1)$

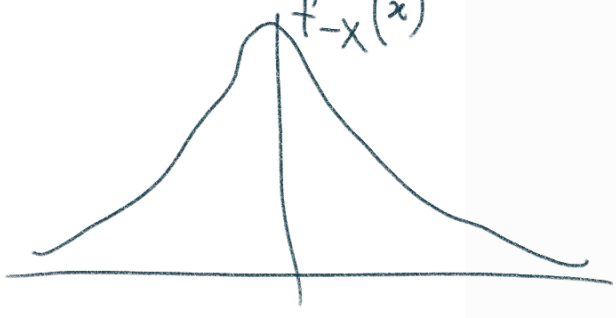
$Z=1 \Rightarrow Y=X \Rightarrow N(0,1)$

Find the PDF of $Y = ZX$.

$f_Y(y)$

$$f_Y(y) = f_{Y|Z}(y|z=-1)\,P(Z=-1) + f_{Y|Z}(y|z=+1)\,P(Z=+1)$$

$(X) \cdot -1 \Rightarrow$



$$f_Y(y) = f_X(x)$$

CDF:

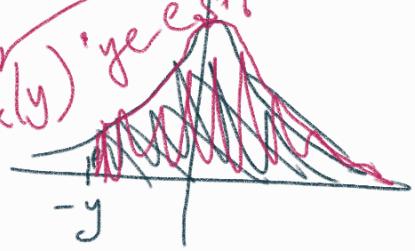
$$P(Y \le y) = P(X \le y, Z=+1) + P(X > -y, Z=-1)$$

$$= P(X \le y)\,P(Z=1) + P(X > y)\,P(Z=-1)$$

($P(Z=1)$: p; $P(Z=-1)$: $1-p$)

$$= p\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx + (1-p)\int_{-y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

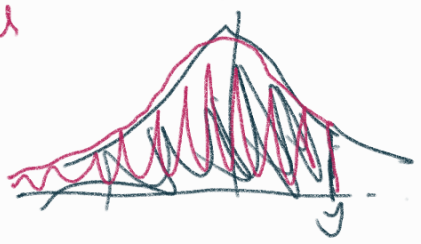
$F_X(y)$'ye eşit



$$F_Y(y) = p\,F_X(y) + (1-p)\,F_X(y)$$

$F_Y(y) = F_X(y)$. Y ve X aynı dağılımlı

o zaman $f_Y(y) = f_X(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

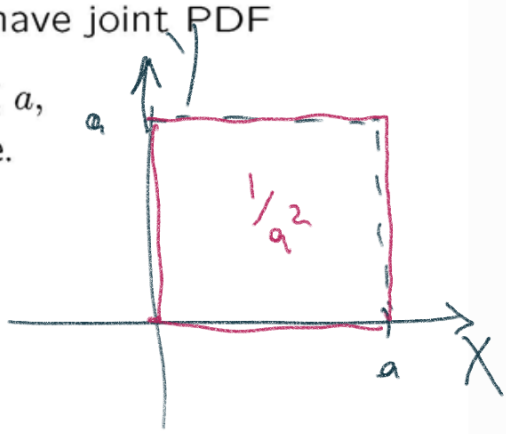


$$W(\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Problem 6.4.13

For a constant $a > 0$, random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/a^2 & 0 \le x, y \le a, \\ 0 & \text{otherwise.} \end{cases}$$



Find the CDF and PDF of random variable

$$\longrightarrow W = \max\left(\frac{X}{Y}, \frac{Y}{X}\right).$$

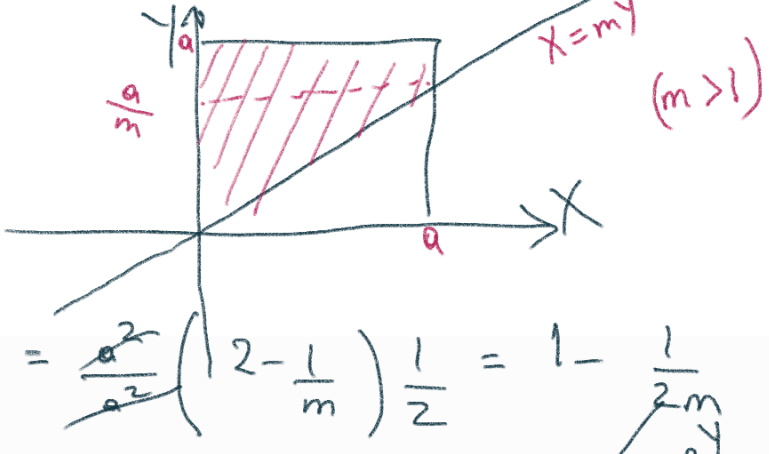
Hint: Is it possible to observe $W < 1$? (HAYIR)

$\infty > W \ge 1$

$M = \frac{X}{Y}$ $\quad N = \frac{Y}{X}$

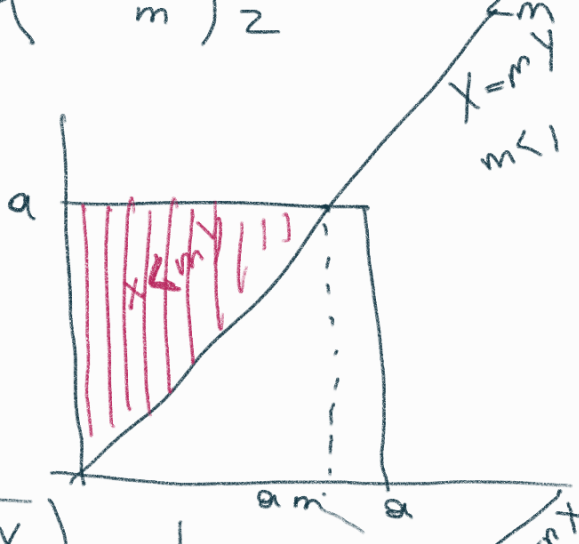
$$P(M \le m) = P\left(\frac{X}{Y} \le m\right)$$

$$P(X \le mY)$$



$m > 1$: $\frac{1}{a^2}\left(\left(a + a - \frac{a}{m}\right)\frac{a}{2}\right) = \frac{a^2}{a^2}\left(2 - \frac{1}{m}\right)\frac{1}{2} = 1 - \frac{1}{2m}$

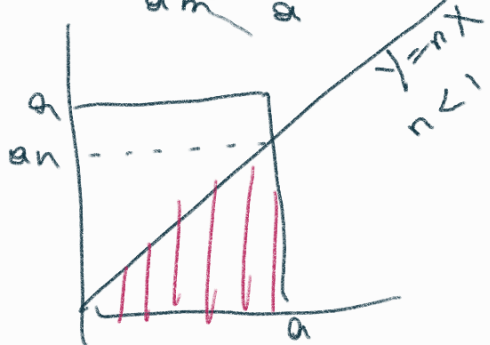
$m < 1$: $\frac{a \times am}{2}\frac{1}{a^2} = \frac{m}{2}$



$$F_M(m) = \begin{cases} \frac{m}{2} & 0 < m < 1 \\ 1 - \frac{1}{2m} & 1 < m \end{cases}$$

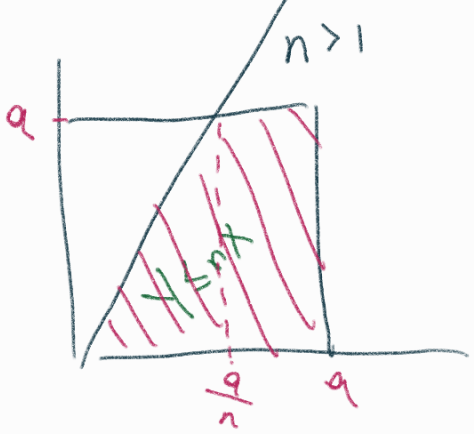
$$P(N \le n) = P\left(\frac{Y}{X} \le n\right) = P(Y \le nX)$$

$$= \frac{a \cdot an}{2a^2} = \frac{n}{2}$$



$n>1$: $P(N \le n) = \dfrac{\left(a + a - \frac{a}{n}\right) a}{2a^2}$

$= \left(2 - \frac{1}{n}\right)\frac{1}{2} = 1 - \frac{1}{2n}$



$$P(N<n) = \begin{cases} \frac{n}{2} & 0 \le n < 1 \\ 1 - \frac{1}{2n} & n > 1 \end{cases}$$

$W = \max\left(M, \frac{1}{M}\right)$

$P(W \le w) = P\left(M \le w, \frac{1}{M} \le w\right)$

$W>1$ $= P\left(M \le w, \frac{1}{M} \le w\right)$

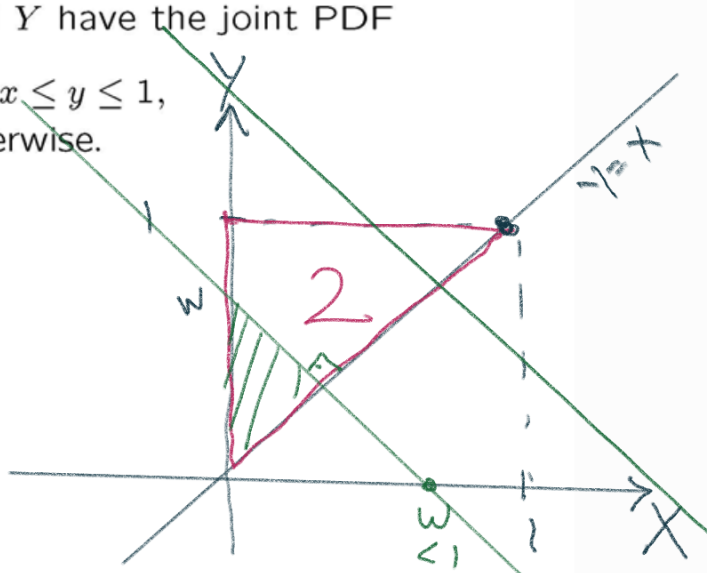
$F_W(w) = 1 - \frac{1}{2w}$, $w>1$

$f_W(w) = \frac{1}{2w^2}$, $w>1$

Problem 6.5.3

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

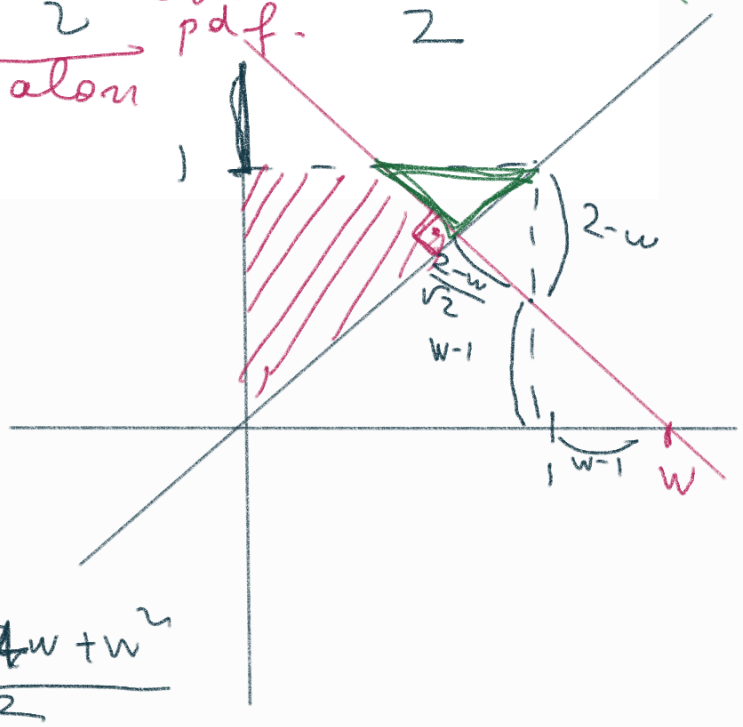


$P(W \le w) = P(X+Y \le w)$

$0 \le W \le 2 \Rightarrow S_W = [0,2]$

- $0 \le w \le 1$: Taralı alana göre:

$$P(W \le w) = \underbrace{\left(\frac{w}{\sqrt{2}}\right)^2 \frac{1}{2}}_{\text{üçgen alan}} \times \underbrace{2}_{\text{pdf.}} = \frac{w^2}{2}$$



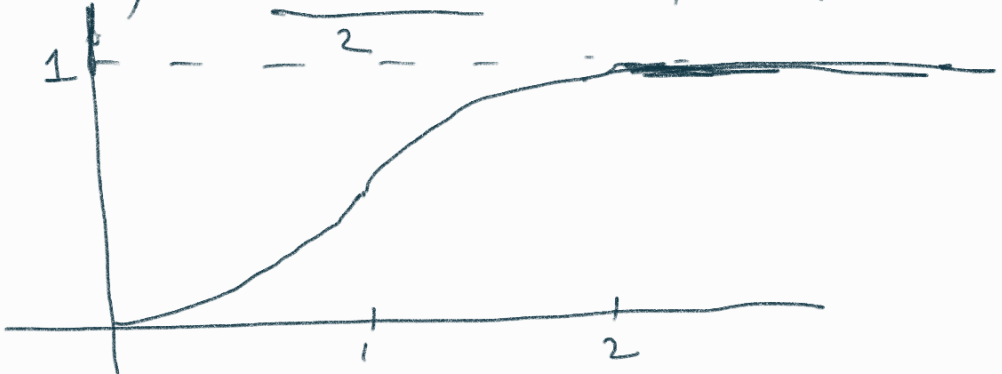
- $1 \le w \le 2$:

$$P(W \le w) = \left(\frac{1}{2} - \left(\frac{2-w}{\sqrt{2}}\right)^2 \frac{1}{2}\right) 2$$

$$1 - \frac{(2-w)^2}{2} = 1 - \frac{4 - 4w + w^2}{2}$$

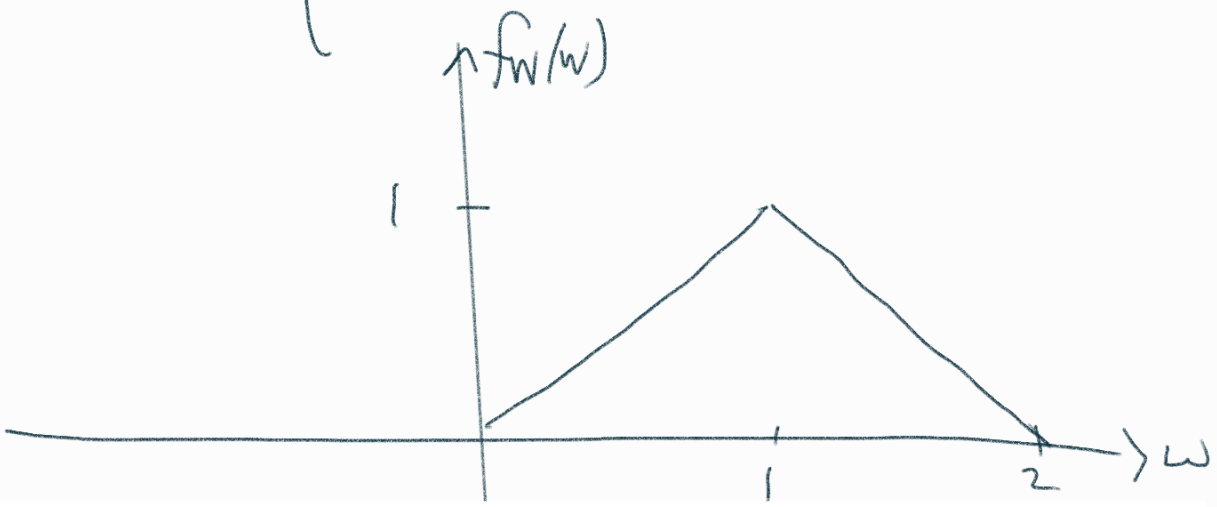
$$= \frac{2 - 4 + 4w - w^2}{2}$$

$$P(W \le w) = \frac{4w - w^2 - 2}{2} \qquad 1 \le w \le 2$$



$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \le w \le 1 \\ \frac{4w - w^2 - 2}{2} & 1 \le w \le 2 \\ 1 & 2 < w \end{cases}$$

$$f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{o.w.} \end{cases}$$



Problem 6.5.5

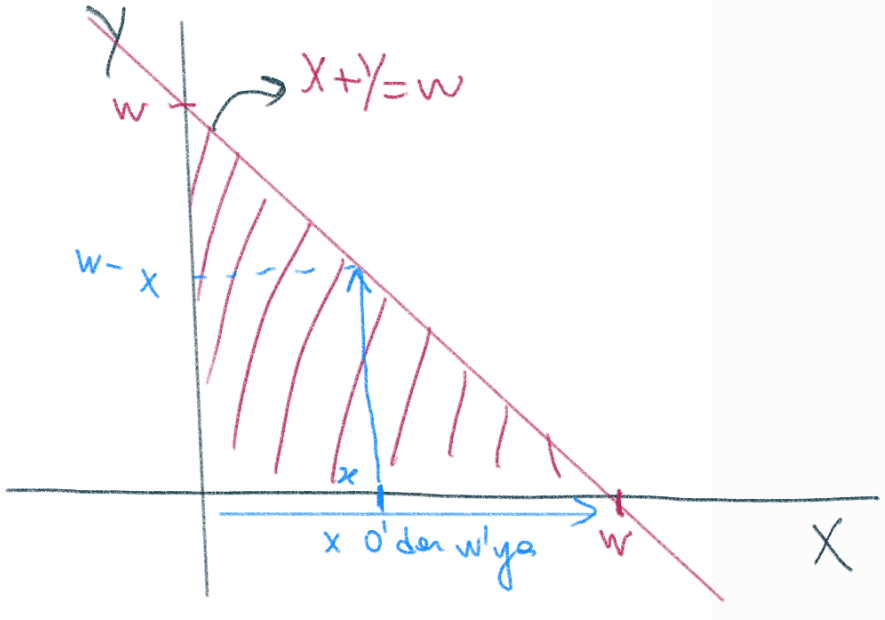
Random variables X and Y are independent exponential random variables with expected values $E[X] = 1/\lambda$ and $E[Y] = 1/\mu$. If $\mu \neq \lambda$, what is the PDF of $W = X + Y$? If $\mu = \lambda$, what is $f_W(w)$?

$X \ge 0$

$Y \ge 0$

$F_W(w) = P(W \le w)$

$= P(X + Y \le w)$



$$\int_{x=0}^{w} \int_{y=0}^{w-x} f_X(x) f_Y(y) \, dy \, dx$$

$$= \int_{x=0}^{w} \int_{y=0}^{w-x} \lambda e^{-\lambda x} \mu e^{-\mu y} \, dy \, dx$$

$$= \int_{x=0}^{w} \lambda e^{-\lambda x} \underbrace{\int_{y=0}^{w-x} \mu e^{-\mu y} \, dy}_{} \, dx$$

$$\int_{x=0}^{w} \lambda e^{-\lambda x} \left(1 - e^{-\mu(w-x)}\right) dx$$

$$= \int \lambda e^{-\lambda x} - \lambda e^{-\mu w} e^{-\lambda x} e^{\mu x} \, dx = \int_0^w \lambda e^{-\lambda x} - \lambda e^{-\mu w} e^{-(\lambda-\mu)x} \, dx$$

$$= -e^{-\lambda x} + \frac{e^{-\mu w} \lambda}{\lambda-\mu} e^{-(\lambda-\mu)x} \Bigg|_0^w$$

$$= -e^{-\lambda w} + \frac{e^{-\mu w} \lambda}{\lambda-\mu} e^{-(\lambda-\mu)w} + 1 - \frac{e^{-\mu w} \lambda}{\lambda-\mu}$$

$$F_W(w) = \left(1 - e^{-\lambda w}\right) - \frac{e^{-\mu w} \lambda}{\lambda-\mu} \left(1 - e^{-(\lambda-\mu)w}\right)$$

$\mu = \lambda$ olsaydı:

$F_W(w) = 1 - e^{-\lambda w}$ $\Rightarrow$ exponential

$$= -e^{-\lambda w} + \frac{\lambda}{\lambda-\mu} e^{-\lambda w} + 1 - \frac{\lambda}{\lambda-\mu} e^{-\mu w}$$

$$= \frac{\mu}{\lambda-\mu} e^{-\lambda w} - \frac{\mu}{\lambda-\mu} e$$

$$\int_0^w f_X(x) f_Y(w-x)\, dx = f_W(w)$$

$$= \int_0^w \lambda e^{-\lambda x} \mu e^{-\mu(w-x)} dx$$

$$= \int_0^w \lambda\mu e^{-\lambda x} e^{-\mu w} e^{\mu x} dx = \int_0^w \lambda\mu e^{-\mu w} e^{-(\lambda+\mu)x} dx$$

$$= \frac{-\lambda\mu e^{-\mu w}}{\lambda+\mu} e^{-(\lambda+\mu)x} \Big|_0^w$$

$$= \left(1 - e^{-(\lambda+\mu)w}\right) \frac{\lambda\mu}{\lambda+\mu} e^{-\mu w}$$

$$= \left(e^{-\mu w} - e^{-\lambda w}\right)\left(\frac{\lambda\mu}{\lambda+\mu}\right)$$

$\lambda = \mu$ için

$$\int_0^w \lambda^2 e^{-\lambda x} e^{-\lambda(w-x)} dx = \int_0^w \lambda^2 e^{-\lambda w} dx$$

$$= \lambda^2 w e^{-\lambda w}$$

Erlang $(2, \lambda)$

2 üstel r.d.'in toplamı Erlang 2 dağılımı.

5.3.6

$$P_{N,K}(n,k) = \begin{cases} \dfrac{100^n e^{-100}}{(n+1)!} & k = 0, 1, \ldots, n \\ 0 & \text{o.w} \end{cases} \qquad n = 0, 1, \ldots$$

$P_N(n) = ?$

$$P_N(n) = \sum_{k=0}^{n} P_{N,K}(n,k)$$

$$P_N(n) = \frac{100^n e^{-100}}{n!} \quad \text{Poisson}(100)$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \sum_{n=0}^{\infty} \frac{100^n e^{-100}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{100} \frac{100^{n+1} e^{-100}}{(n+1)!}$$

$$P_K(k) = \frac{1}{100}\left[1 - e^{-100}\right]$$

Problem 7.1.5

Every day you consider going jogging. Before each mile, including the first, you will quit with probability q, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let M equal the number of miles that you run on an arbitrary day. Find the PMF $P_M(m)$.

(b) Let r be the probability that you run a marathon on an arbitrary day. Find r.

(c) Let J be the number of days in one year (not a leap year) in which you run a marathon. Find the PMF $P_J(j)$. This answer may be expressed in terms of r found in part (b).

(d) Define $K = M - 26$. Let A be the event that you have run a marathon. Find $P_{K|A}(k)$.

a) $P_M(m) = (1-q)^m q$, $m = 0, 1, \ldots$

b) $P(\text{maraton}) = P(M \geq 26)$

$$= \sum_{m=26}^{\infty} (1-q)^m q$$

$$= (1-q)^{26} q \sum_{i=0}^{\infty} (1-q)^i$$

$$= (1-q)^{26} q \frac{1}{1-(1-q)} = (1-q)^{26} = r$$

c) $P_J(j) = \binom{365}{j} r^j (1-r)^{365-j}$, $j = 0, 1, \ldots 365$

Binom

d) $$P_{K|A}(k) = \begin{cases} \dfrac{P_K(k)}{P(A)}, & \text{if } k \in A \\ 0 & \text{o.w.} \end{cases}$$

→ A: Maraton koştuk. ⇒→ $M \geq 26$ ⇒ r olasılıkla → $P(A) = r$

→ $K = M - 26$ → $P_K(k) = P_M(k+26)$, $k = 0, 1, \ldots$

$$P_{K|A}(k) = \begin{cases} \dfrac{P_K(k)}{r} & k = 0, 1, \ldots, \infty \\ 0 & \text{o.w.} \end{cases}$$

$$= P_{K|A}(k) = \begin{cases} \dfrac{P_M(k+26)}{r} & k = 0, 1, \ldots, \infty \\ 0 & \text{o.w.} \end{cases}$$

$$\frac{(1-q)^{k+26}\, q}{(1-q)^{26}} = (1-q)^k q$$

$$= P_{K|A}(k) = \begin{cases} (1-q)^k q & k = 0, 1, \ldots, \infty \\ 0 & \text{o.w.} \end{cases}$$

Problem 7.1.7

A test for diabetes is a measurement X of a person's blood sugar level following an overnight fast. For a healthy person, a blood sugar level X in the range of $70-110$ mg/dl is considered normal. When a measurement X is used as a test for diabetes, the result is called positive (event T^+) if $X \geq 140$; the test is negative (event T^-) if $X \leq 110$, and the test is ambiguous (event T^0) if $110 < X < 140$.

Given that a person is healthy (event H), a blood sugar measurement X is the Gaussian $(90, 20)$ random variable. Given that a person has diabetes, (event D), X is the Gaussian $(160, 40)$ random variable. A randomly chosen person is healthy with probability $P[H] = 0.9$ or has diabetes with probability $P[D] = 0.1$.

(a) What is the conditional PDF $f_{X|H}(x)$?

(b) Calculate the conditional probabilities $P[T^+|H]$, and $P[T^-|H]$.

(c) Find $P[H|T^-]$, the conditional probability that a person is healthy given the event of a negative test.

(d) When a person has an ambiguous test result (T^0), the test is repeated, possibly many times, until either a positive T^+ or negative T^- result is obtained. Let N denote the number of times the test is given. Assuming that for a given person the result of each test is independent of the result of all other tests, find the conditional PMF of N given event H that a person is healthy. Note that $N = 1$ if the person has a positive T^+ or negative T^- result on the first test.

$\rightarrow f_{X|H}(x) = \mathcal{N}(90, 20)$

$f_{X|D}(x) = \mathcal{N}(160, 40)$

$P[H] = 0.9 \qquad P[D] = 0.1$

a) $f_{X|H}(x) = \frac{1}{\sqrt{2\pi}\,20} e^{-(x-90)^2/800}, \quad -\infty < x < \infty$

b) $P[T^+|H] = \int_{140}^{\infty} f_{X|H}(x)\,dx$

$f_{X|H} \rightarrow \mathcal{N}(90, 20)$

$P(X > 140) = P\left(\frac{X-90}{20} > \frac{140-90}{20}\right)$

$\frac{140-90}{20} = 2.5$

$= Q(2.5)$

$= 0.0062$

$$P(T^-|H) = \int_{-\infty}^{110} f_{x|H}(x)\,dx$$

$f_{x|H} \hookrightarrow N(90, 20)$

$$= P(X < 110) = P\left(\frac{X-90}{20} < \frac{110-90}{20}\right)$$

$$= 1 - Q(1)$$

$$= 0.84$$

c) $$P(H|T^-) = \frac{P(T^-|H)P(H)}{P(T^-)}$$

$$= \frac{P(\underset{0.84}{T^-}|H)\,P(H)_{0.9}}{P(\underset{0.84}{T^-}|H)\,P(H) + P(\underset{0.1056}{T^-}|D)\,P(D)_{0.1}}$$

$$P(T^-|D) = \int_{-\infty}^{110} f_{x|D}(x)\,dx$$

$f_{x|D} \hookrightarrow N(160, 40)$

$$P(X \le 110) = P\left(\frac{X-160}{40} \le \frac{110-160}{40}\right)$$

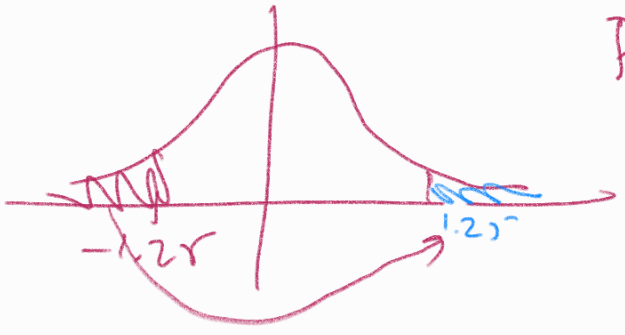
$$= P(Z \le -1.25)$$

$$= Q(1.25)$$

$$= 1 - \phi(1.25)$$

$$= 1 - 0.8944$$

$$= 0.1056$$



$$= \frac{0.84 \times 0.9}{0.84 \times 0.9 + 0.1056 \times 0.1}$$

$$= \frac{0.756}{0.756 + 0.011} = \frac{756}{767}$$

$$= 0.9861$$

d) $P(A) = P(110 \le X \le 140)$

↳ ambiguous

$P(A|H) = P(110 \le X \le 140 \,|\, H)$

↳ $\mathcal{N}(90, 20)$

$= P\left(\frac{20}{20} \le \frac{X-90}{20} \le \frac{50}{20}\right)$

$\frac{X-90}{20} = Z \sim N(0,1)$

$P(1 \le z \le 2.5)$

$= Q(1) - Q(2.5)$

$= \phi(2.5) - \phi(1)$

$= 0.993790 - 0.841300$

$P(A|H) = 0.15249 = p$

$P_N(n) = p^{n-1}(1-p)$, $n = 1, 2, \ldots$

$N \sim$ Geometrik$(1-p)$

$E[N] = \frac{1}{1-p} \cong \frac{1}{0.85}$

Problem 7.2.4

Y is the exponential (0.2) random variable. Given $A = \{Y < 2\}$, find:
(a) $f_{Y|A}(y)$,
(b) $\mathrm{E}[Y|A]$.

$$f_Y(y) = 0.2\, e^{-0.2y} \qquad y > 0$$

a)
$$f_{Y|A}(y) = \begin{cases} \dfrac{f_Y(y)}{P(A)} & y \in A \\ 0 & \text{o.w} \end{cases}$$

$$P(A) = P(Y<2) = \int_0^2 0.2\, e^{-0.2y}\, dy$$

$$= -e^{-0.2y}\Big|_0^2 = 1 - e^{-0.4}$$

$$f_{Y|A}(y) = \begin{cases} \dfrac{0.2\, e^{-0.2y}}{1-e^{-0.4}} & 0 \le y \le 2 \\ 0 & \text{o.w} \end{cases}$$

$$E[Y|A] = \int_{y \in A} y\, f_{Y|A}(y)\, dy$$

$$= \int_0^2 \frac{y\, 0.2 e^{-0.2y}}{1-e^{-0.4}}\, dy$$

$$\left(\int u\,dv = uv\Big| - \int v\,du\right)$$

$u = y \qquad dv = 0.2e^{-0.2y}\,dy$

$du = dy \qquad v = -e^{-0.2y}$

$$= \frac{1}{1-e^{-0.4}}\left[-y\,e^{-0.2y}\Big|_0^2 - \int_0^2 -e^{-0.2y}\,dy\right]$$

$$= \frac{1}{1-e^{-0.4}}\left[-2e^{-0.4} + \frac{-1}{0.2}e^{-0.2y}\Big|_0^2\right]$$

$$= \frac{1}{1-e^{-0.4}}\left[-2e^{-0.4} - \frac{1}{0.2}e^{-0.4} + \frac{1}{0.2}\right]$$

Problem 7.2.6

The number of pages X in a document has PMF

$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4, \\ 0.1 & x = 5,6,7,8, \\ 0 & \text{otherwise.} \end{cases}$$

A firm sends all documents with an even number of pages to printer A and all documents with an odd number of pages to printer B.

(a) Find the conditional PMF of the length X of a document, given the document was sent to A. What are the conditional expected length and standard deviation?

(b) Find the conditional PMF of the length X of a document, given the document was sent to B and had no more than six pages. What are the conditional expected length and standard deviation?

Or: 1,3,5,7 sayfa → B, 2,4,6,8 sayfa → A

$$P_{X|A}(x) = \begin{cases} \dfrac{P_X(x)}{P(A)}, & x \in A \\ 0 & \text{O.W} \end{cases}$$

a) $P(A) = 0.15 + 0.15 + 0.1 + 0.1 = 0.5$

$$P_{X|A}(x) = \begin{cases} \frac{0.15}{0.5} = 0.3 & x = 2, 4, \\ \frac{0.1}{0.5} = 0.2 & x = 6, 8 \\ 0 & o.w \end{cases}$$

b) $\{B \cap (X \leq 6)\} = C = \{1, 3, 5\}$

$$P(C) = P_X(1) + P_X(3) + P_X(5) = 0.4$$

$$P_{X|C}(x) = \begin{cases} \frac{0.15}{0.4} = 3/8 & x = 1, 3 \\ \frac{0.1}{0.4} = 2/8 & x = 5 \\ 0 & o.w \end{cases}$$

$$E[X|C] = 1 \times 3/8 + 3 \times 3/8 + 5 \times 2/8 = \frac{22}{8} = \frac{11}{4}$$

$$Var(X|C) = E[X^2|C] - (E[X|C])^2$$

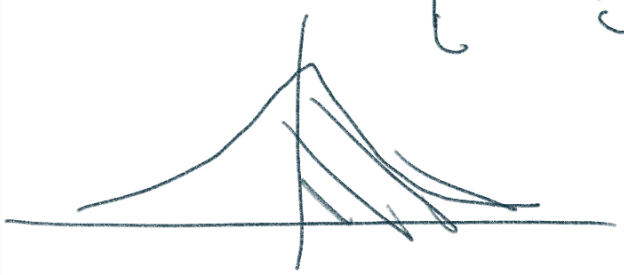
$$E[X^2|C] = 1^2 \times 3/8 + 3^2 \times \frac{3}{8} + 5^2 \times \frac{2}{8} = \frac{80}{8} = 10$$

$$Var(X|C) = 10 - \left(\frac{11}{4}\right)^2 = 10 - \frac{121}{16} \simeq 2.44$$

Problem 7.2.8

W is the Gaussian $(0,4)$ random variable. Given the event $C = \{W > 0\}$, find the conditional PDF, $f_{W|C}(w)$, the conditional expected value, $\mathrm{E}[W|C]$, and the conditional variance, $\mathrm{Var}[W|C]$.

$$f_{W|C}(w) = \begin{cases} \dfrac{f_W(w)}{P(C)} & w \in C \\ 0 & \text{o.w} \end{cases}$$



$$P(C) = 1/2$$

$$f_{W|C}(w) = \begin{cases} 2\,\dfrac{1}{\sqrt{2\pi}\,4}\, e^{-w^2/32} & w > 0 \\ 0 & \text{o.w} \end{cases}$$

$$E[W|C] = \int_0^\infty 2w\,\frac{1}{\sqrt{2\pi}\,4}\, e^{-w^2/32}\, dw$$

$w^2 = u \qquad 2w\,dw = du$

$w : 0 \to \infty$

$u : 0 \to \infty$

$$= \int_{u=0}^{\infty} \frac{1}{4\sqrt{2\pi}}\, e^{-u/32}\, du = \frac{-32}{4\sqrt{2\pi}}\, e^{-u/32}\Big|_0^\infty$$

$$E[W|C] = \frac{8}{\sqrt{2\pi}}$$

$$E[W^2|C] = \int_{w=0}^{\infty} 2w^2 \frac{1}{4\sqrt{2\pi}} e^{-w^2/32}\, dw$$

$$= \int_{w=-\infty}^{\infty} w^2 \underbrace{\frac{1}{4\sqrt{2\pi}} e^{-w^2/32}}_{N(0,4)}\, dw = 16$$

0 ortalamalı, 4 std sapmalı Gauss rast. değişkenin varyansı (karesinin beklenen değeri)

$$Var(W|C) = E[W^2|C] - E[W|C]^2$$

$$= 16 - \frac{64}{2\pi^2} = 16 - \frac{32}{\pi^2} = 12.75$$

Problem 7.2.9

The time between telephone calls at a telephone switch is the exponential random variable T with expected value 0.01. $\longrightarrow \lambda = 10$

(a) What is $E[T|T > 0.02]$, the conditional expected value of T?

(b) What is $\mathrm{Var}[T|T > 0.02]$, the conditional variance of T?

$$f_T(t) = 10e^{-10t}, \quad t \geq 0$$

a) $A = \{T > 0.02\} \Rightarrow P(A) = \int_{0.02}^{\infty} 10e^{-10t}\, dt = e^{-0.2}$

$$f_{T|A}(t) = \begin{cases} \dfrac{10e^{-10t}}{e^{-0.2}}, & t > 0.02 \\ 0 & \text{o.w} \end{cases}$$

$$E[T|A] = \int_{0.02}^{\infty} t \, \frac{10e^{-10t}}{e^{-0.2}} \, dt$$

$$\left(\int u\,dv = uv \Big| - \int v\,du \right)$$

$$\boxed{\begin{array}{ll} t = u & 10e^{-10t}\,dt = dv \\ dt = du & V = -e^{-10t} \end{array}}$$

$$= \left[-t e^{-10t} \Big|_{0.02}^{\infty} - \int_{0.02}^{\infty} -e^{-10t}\,dt \right] \frac{1}{e^{-0.2}}$$

$$= \left[0.02e^{-0.2} + \frac{-1}{10} e^{-10t} \Big|_{0.02}^{\infty} \right] \frac{1}{e^{-0.2}}$$

$$= \left[0.02e^{-0.2} + \frac{1}{10} e^{-0.2} \right] \frac{1}{e^{-0.2}} = \frac{e^{-0.2}(0.12)}{e^{-0.2}} = 0.12 //$$

Not: $0.12 = \underbrace{0.02}_{T>0.02} + \underbrace{0.1}_{1/\lambda}$

HAFIZASIZ
MEMORYLESS
üstel dağılımın özelliği

b) $Var(T \mid T>0.02)$

$$= E[T^2 / T>0.02) + E[T \mid T>0.02]^2$$

$$E[T^2 | T>0.02] = \int_{0.02}^{\infty} t^2 \, \frac{10e^{-10t}}{e^{-0.2}} \, dt$$

2 defa kısmi integral.

Problem 7.2.10

As the final rider in the final 60 km time trial of the Tour de France, Roy must finish in time $T \leq 1$ hour to win the Tour. He has the choice of bike made of (1) carbon fiber or (2) titanium. On the carbon fiber bike, his speed V over the course is the continuous uniform random variable with $E[V] = 58$ km/hr and $\mathrm{Var}[V] = 12$. On the titanium bike, V is the exponential random variable with $E[V] = 60$ km/hr. $f_{V|Titan}(v) = \frac{1}{60} e^{-v/60}$

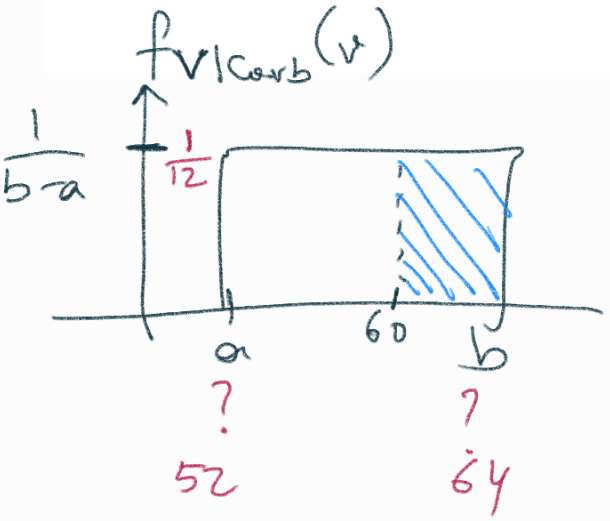
(a) Roy chooses his bike to maximize $P[W]$, the probability he wins the Tour. Which bike does Roy choose and what is $P[W]$?

(b) Suppose instead that Roy flips a fair coin to choose his bike. What is $P[W]$?

$P(T<1 \mid Carb) \gtrless P(T<1 \mid Titan)$

$P(T<1 \mid Carb) = 1/3$

$P(T<1 \mid Titan) = e^{-1} = \frac{1}{2.718}$



$\frac{a+b}{2} = 58$

$\frac{(b-a)^2}{12} = 12$

$a+b = 116$

$b-a = 12$

$2b = 128$

$b = 64$

$a = 52$

$P(T<1) = P\left(\frac{60}{V} < 1\right)$

$= P(V > 60)$

$P(V>60 \mid Carb) = \frac{1}{3}$

$$P(V>60 \mid \text{Titan}) = \int_{60}^{\infty} \frac{1}{60} e^{-t/60}\, dt$$

$$= -e^{-t/60}\Big|_{60}^{\infty} = e^{-1}$$

Titanyum daha iyi ($\frac{1}{2.718}$ kazanma şansı)

b)

$$P(V>60) = P(V>60 \mid \text{Corb})\,P(\text{Corb}) + P(V>60 \mid \text{Titan})\,P(\text{Titan})$$

$P(\text{Corb}) \to 1/2$, $P(\text{Titan}) \to 1/2$, $P(V>60 \mid \text{Corb}) = \frac{1}{3}$, $P(V>60 \mid \text{Titan}) = \frac{1}{2.718}$

$$P(V>60) = \left(\frac{1}{3} + \frac{1}{2.718}\right) 0.5$$

Problem 7.3.1

X and Y are independent identical discrete uniform $(1, 10)$ random variables. Let A denote the event that $\min(X, Y) > 5$. Find the conditional PMF $P_{X,Y|A}(x, y)$.

Sürekli ise

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

($f_X(x) = 1/9$, $f_Y(y) = 1/9$)

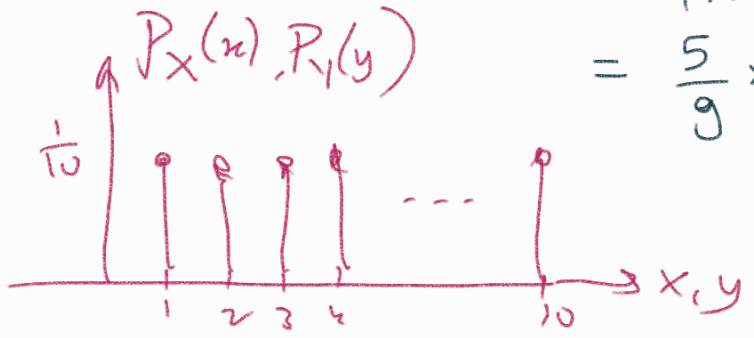
$$= \begin{cases} \frac{1}{81} & 1 < x, y < 10 \\ 0 & \text{o.w} \end{cases}$$

$$f_{XY|A}(x,y) = \begin{cases} \dfrac{\frac{1}{81}}{P(A)} = \frac{1}{25} & 1<x,y<10,\ \min(x,y)>5 \\ 0 & \text{o.w} \end{cases}$$

$5<x,y<10$

$$P(A) = P(\min(X,Y)>5) = P(X>5)P(Y>5) = \frac{5}{9}\times\frac{5}{9} = \frac{25}{81}$$

Ayrik:



$$P_{XY}(x,y) = \begin{cases} \frac{1}{100} & x,y \in \{1,\dots,10\} \\ 0 & \text{o.w} \end{cases}$$

$x,y \in \{6,7,8,9,10\}$

$$P_{XY|A}(x,y) = \begin{cases} \dfrac{\frac{1}{100}}{P(A)} = \frac{1}{25} & x,y \in \{1,\dots,10\} \text{ ve } \min(x,y) \in \{6,7,8,9,10\} \\ 0 & \text{o.w} \end{cases}$$

$$P(A) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

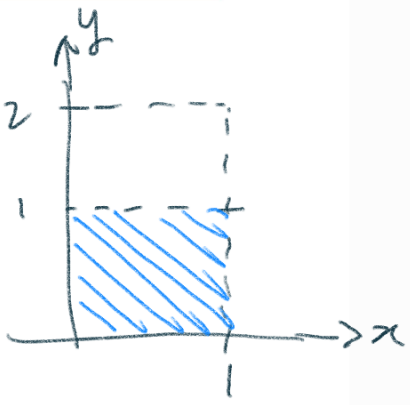
$$P(A) = P(\min(X,Y)>5) = P(X,Y>5) = P(X>5)P(Y>5)$$

$\frac{1}{2}$ $\frac{1}{2}$

Problem 7.3.5

X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1; \\ & 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$



Let $A = \{Y \le 1\}$.

(a) What is P[A]?

(b) Find $f_{X,Y|A}(x,y)$.

(c) Find $f_{X|A}(x)$ and $f_{Y|A}(y)$.

a)

$$P[A] = \int_{y=0}^{1}\int_{x=0}^{1} f_{XY}(x,y)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1}\left(\frac{x}{3}+\frac{y}{3}\right)dx\,dy$$

$$= \int_{y=0}^{1}\left(\frac{x^2}{6}+\frac{xy}{3}\Big|_0^1\right)dy$$

$$= \int_0^1 \frac{1}{6}+\frac{y}{3}\,dy = \frac{y}{6}+\frac{y^2}{6}\Big|_0^1 = \frac{1}{6}+\frac{1}{6} = \boxed{\frac{1}{3}}$$

b)

$$f_{XY|A}(x,y) = \begin{cases} \dfrac{f_{XY}(x,y)}{P(A)} & 0<x<1 \\ & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

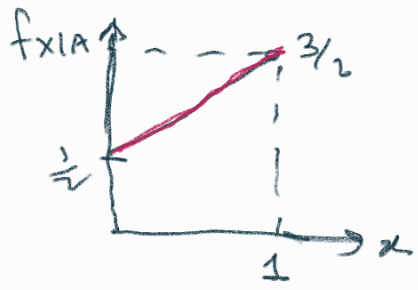
$$f_{XY|A}(x,y) = \begin{cases} x+y & 0<x<1 \\ & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

c)

$$f_{X|A}(x) = \int_{y=0}^{1} f_{XY|A}(x,y)\,dy$$

$$= \int_{y=0}^{1} x+y\,dy = xy+\frac{y^2}{2}\Big|_0^1 = x+\frac{1}{2}$$

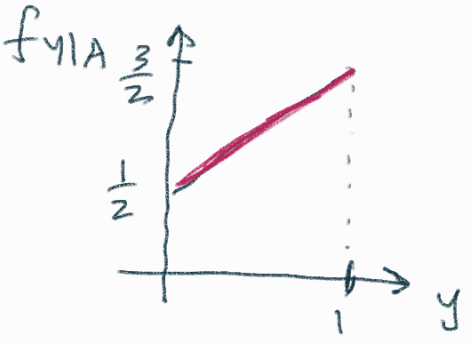
$$\boxed{f_{X|A}(x) = x+\frac{1}{2},\quad 0 \le x \le 1}$$



$$f_{Y|A}(y) = \int_{x=0}^{1} f_{XY|A}(x,y)\,dx$$

$$= \int_{x=0}^{1} x+y \; dx = \frac{x^2}{2} + xy \Big|_0^1$$

$$= \frac{1}{2} + y$$



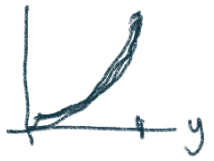
$$\left.\begin{aligned} f_{XY|A} &= x+y \\ f_{X|A} &= \tfrac{1}{2} + x \\ f_{Y|A} &= \tfrac{1}{2} + y \end{aligned}\right\} \quad X, Y \text{ bağımsız değil.}$$

Problem 7.3.9

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$



Let $A = \{X > Y\}$.

(a) What are E[X] and E[Y]?

(b) What are E[$X|A$] and E[$Y|A$]?

a) $$E[X] = \int_0^1 \underbrace{2x}_{pdf} \cdot x\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$E[Y] = \int_0^1 3y^2 \cdot y\,dy = \frac{3y^4}{4}\Big|_0^1 = \frac{3}{4}$$

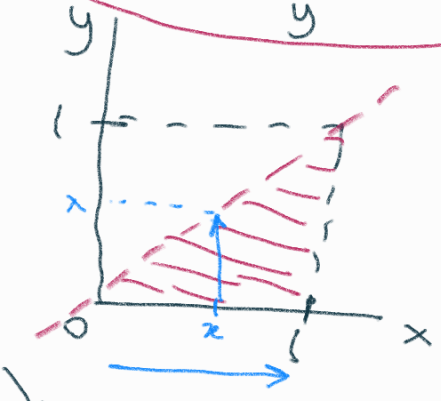
b) $$f_{XY}(x,y) = 6xy^2 \qquad 0 \le x \le 1, \; 0 \le y \le 1$$

$$f_{XY|A}(x,y) = \begin{cases} \dfrac{f_{XY|A}(x,y)}{P(A)} \to 2/5 & 0 \le y \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{XY|A}(x,y) = \begin{cases} 15xy^2 & 0 \le y \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|A}(x) = \int_0^x 15xy^2\,dy = 5xy^3\Big|_0^x = 5x^4, \quad 0 \le x \le 1$$

$$f_{Y|A}(y) = \int_y^1 15xy^2\,dx = \frac{15x^2y^2}{2}\Big|_y^1 = \frac{15y^2}{2} - \frac{15y^4}{2}, \quad 0 \le y \le 1$$



$$\int_0^1 5x^4 \cdot x\,dx = \frac{5x^6}{6}\Big|_0^1 = 5/6$$

$$E[Y|A] = \int_0^1 y\left(\frac{15y^2}{2} - \frac{15y^4}{2}\right)dy = \frac{15y^4}{8} - \frac{15y^6}{12}\Big|_0^1$$

$$= \frac{15}{8} - \frac{15}{12} = \frac{15}{24} = \frac{5}{8}$$

$$E[Y|A] = \int_{x=0}^1 \int_{y=0}^x y\, f_{XY|A}(x,y)\,dy\,dx$$

$$\int_{x=0}^1 \int_{y=0}^x 15xy^3\,dy\,dx = \int_{x=0}^1 \left(\frac{15xy^4}{4}\Big|_0^x\right)dx$$

$$= \int_0^1 \frac{15x^5}{4}dx = \frac{15x^6}{24}\Big|_0^1$$

$$= \frac{15}{24} = \frac{5}{8}$$

Problem 7.4.8

$Y = ZX$ where X is the Gaussian $(0,1)$ random variable and Z, independent of X, has PMF

$$P_Z(z) = \begin{cases} 1-p & z=-1, \\ p & z=1. \end{cases}$$

True or False:

(a) Y and Z are independent. EVET

(b) Y and X are independent. ~~HAYIR~~

$f_{Y|Z}(y|z) \stackrel{?}{=} f_Y(z)$

a) $f_{Y|Z=1}$ $\quad Y = ZX$

$$P(Y \le y \mid Z=1) = P(XZ \le y \mid Z=1) \xrightarrow{\frac{d}{dx}} f_X(y) = f_{Y|Z=1}(y)$$

$$= P(X \le y \mid Z=1) = F_X(y)$$

$$P(Y \le y \mid Z=-1) = P(XZ \le y \mid Z=-1)$$

$$= P(-X \le y) = P(X \ge -y)$$

$$= 1 - P(X \le -y)$$

$$= P(X > y) \xrightarrow{d/dx} \frac{d}{dy}\{1 - F_X(-y)\}$$

$Z=-1$ de olsa $+1$'de olsa Y'nin dağılımı aynı olduğu için $f_{X|Z=-1} = f_X(y)$ // Y ve Z bağımsızdır.

b) Y ve X bağımsız mı?

$$f_Y(y) = \underbrace{f_{Y|Z=1}(y)}_{f_X(y)} \underbrace{P(Z=1)}_{p} + \underbrace{f_{Y|Z=-1}(y)}_{f_X(y)} \underbrace{P(Z=-1)}_{1-p}$$

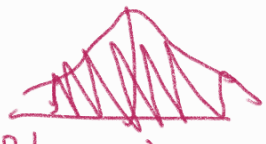
$f_Y(y) = f_X(y) \quad N(0,1)$

$$P(X \le x, Y \le y) = P(X \le x, XZ \le y)$$

$$= P(X \le x, XZ \le y \mid Z=1)\,p + P(X \le x, XZ \le y \mid Z=-1)\,(1-p)$$

$$= P(X \le *, X \le y)\,p + P(X \le x, -X \le y)\,(1-p) \qquad (X \ge -y)$$

$$= P(X \le \min(x,y))\,p + P(-y \le X \le x)\,(1-p) \qquad P(X \ge -y)$$



Kısa Yol: X verildiğinde $Y = ZX$, ya $-x$ olur ya da $+x$ olur.

Normalde $N(0,1)$ dağılımlı olan Y'nin dağılımı, X verildiğinde değişti. Dolayısıyla bunlar bağımsız değildir

Problem 7.4.13

Packets arriving at an Internet router are either voice packets (v) or data packets (d). Each packet is a voice packet with probability p, independent of any other packet. Observe packets at the Internet router until you see two voice packets. Let M equal the number of packets up to and including the first voice packet. Let N equal the number of packets observed up to and including the second voice packet. Find the conditional PMFs $P_{M|N}(m|n)$ and $P_{N|M}(n|m)$. Interpret your results.

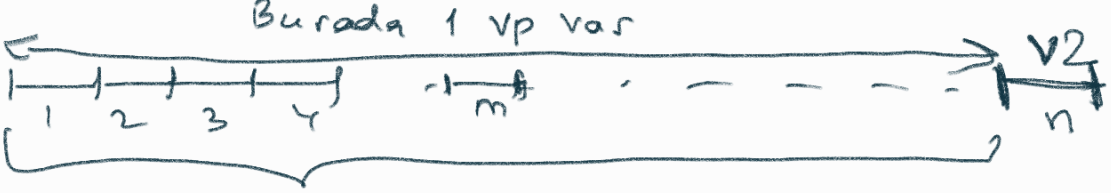
$M \sim \text{Geometric}(p)$ $\qquad P_M(m) = (1-p)^{m-1} p$, $m = 1, \ldots$

$$P_N(n) = \binom{n-1}{1} \cdot (1-p)^{n-2} \cdot p \cdot p = \binom{n-1}{1} (1-p)^{n-2} p^2$$

$$P_{M|N}(m|n) = P(\text{ilk voice paketi } m\text{'inci de} \mid \text{2. voice paketi } n\text{'inci de}) \qquad n = 2, \ldots$$

$= \frac{1}{n-1}$, $m < n$ UNIFORM

Her zaman $N > M$

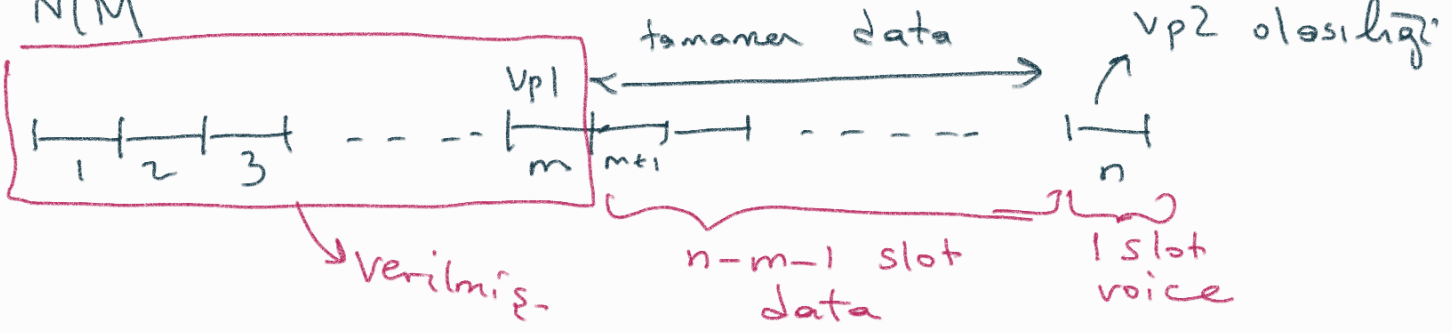


$n-1$

$$P_{M|N}(m|n) = \begin{cases} \frac{1}{n-1} & m = 1, 2, \dots, n-1 \\ 0 & \text{O.W} \end{cases}$$

UNIFORM

$$P_{N|M}(n|m) = P(n\text{'inci de 2. v.p.} \mid m\text{'inci de 1. v.p.})$$



$$P_{N|M}(n|m) = \begin{cases} (1-p)^{n-m-1} p & , \ n > m \\ 0 & \text{O.W.} \end{cases}$$

N-M Geometrik(p)

her v.p geldiğinde süreç yenileniyor.

Problem 7.5.3

The probability model for random variable A is

$$P_A(a) = \begin{cases} 1/3 & a = -1, \\ 2/3 & a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The conditional probability model for random variable B given A is:

$$P_{B|A}(b|-1) = \begin{cases} 1/3 & b = 0, \\ 2/3 & b = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad P_{B|A}(b|1) = \begin{cases} 1/2 & b = 0, \\ 1/2 & b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the probability model for random variables A and B? Write the joint PMF $P_{A,B}(a,b)$ as a table.

(b) If $A = 1$, what is the conditional expected value $\mathrm{E}[B|A = 1]$?

(c) If $B = 1$, what is the conditional PMF $P_{A|B}(a|1)$?

(d) If $B = 1$, what is the conditional variance $\mathrm{Var}[A|B = 1]$ of A?

(e) What is the covariance $\mathrm{Cov}[A, B]$?

a) $P_{B|A}(b|a) = \frac{P_{AB}(a,b)}{P_A(a)} \Rightarrow P_{AB}(-1,0) = \underbrace{P_{B|A}(0|-1)}_{1/3} \underbrace{P_A(-1)}_{1/3}$

A \ B	0	1
-1	1/9	2/9
1	2/6	2/6

$P_{AB}(-1,1) = \underbrace{P_{B|A}(1|-1)}_{2/3} \underbrace{P_A(-1)}_{1/3}$

$P_{AB}(1,0) = \underbrace{P_{B|A}(0|1)}_{1/2} \underbrace{P_A(1)}_{2/3}$

$P_{AB}(1,1) = \underbrace{P_{B|A}(1|1)}_{1/2} \underbrace{P_A(1)}_{2/3}$

b) $E[B|A=1]$

$= \sum_{b=0}^{1} P_{B|A=1}(b)\, b$

$= 0 \cdot \underbrace{P_{B|A=1}(0)}_{1/2} + 1 \cdot \underbrace{P_{B|A=1}(1)}_{1/2}$

$= 1/2$

c) $E[A|B=1]$

$= \sum_{a=-1,1} a \cdot \underbrace{P_{A|B=1}(a)}$

$\frac{P(A=a, B=1)}{P(B=1)} = \frac{P(B=1|A=a)\,P(A=a)}{P(B=1)}$

$= -1 \cdot \frac{P(B=1|A=-1)P(A=-1)}{P(B=1)} + \underbrace{(+1) \cdot \frac{P(B=1|A=1)\,P(A=1)}{P(B=1)}}$

$= \underbrace{(-1) \cdot \frac{2}{3} \cdot \frac{1}{3}}_{5/9} + \underbrace{\frac{(+1) \cdot \frac{1}{2} \cdot \frac{2}{3}}{5/9}}$

$= \frac{\frac{1}{3} - \frac{2}{9}}{5/9} = \frac{1}{5}$

d) $B=1$ ise $\quad Var(A|B=1)$

$= E[A^2|B=1] - \left(E[A|B=1]\right)^2$

$E[A^2|B=1] =$ ($E[A|B=1] = 1/5$)

$$\frac{(-1)^2 \cdot \frac{2}{3} \cdot \frac{1}{3}}{5/9} + \frac{(+1)^2 \cdot \frac{1}{2} \cdot \frac{2}{3}}{5/9} = \frac{\frac{2}{9} + \frac{2}{6}}{5/9}$$

$= 1$ //

$$Var(A|B=1) = 1 - \left(\frac{1}{5}\right)^2$$

$= 24/25$

e) $Cov(A,B) = E[AB] - E[A]E[B]$

$E[AB] = -1 \cdot 0 \cdot \frac{1}{9} + (-1)\cdot(1)\cdot\frac{2}{9} + 1 \cdot 0 \cdot \frac{2}{6} + 1 \cdot 1 \cdot \frac{2}{6}$

$= \frac{2}{6} - \frac{2}{9} = \frac{1}{9}$

$E[A] = -1 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{1}{3}$

$E[B] = 0 \cdot 4/9 + 1 \cdot 5/9 = 5/9$

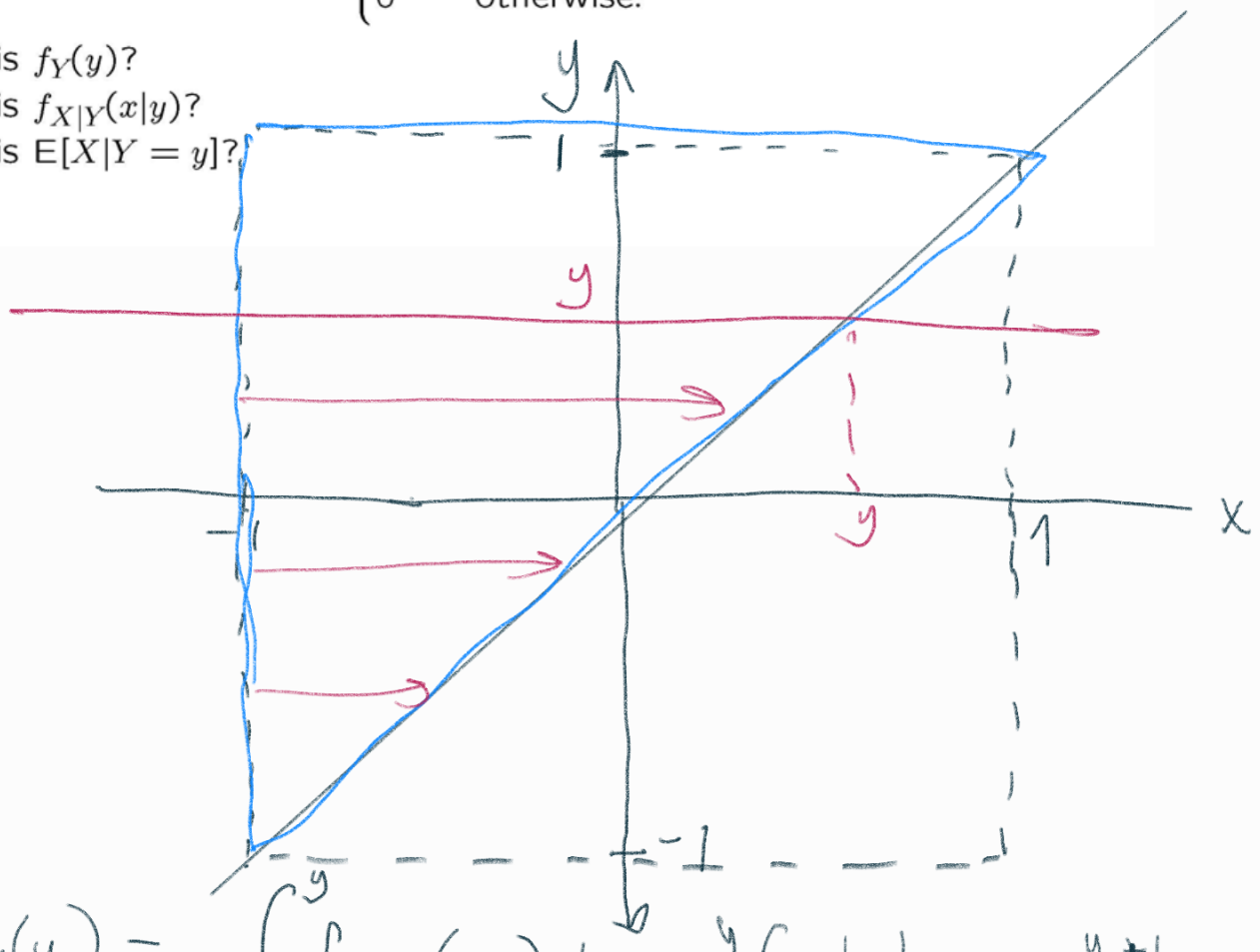
$Cov(A,B) = \frac{1}{9} - \frac{1}{3} \times \frac{5}{9} = \frac{-2}{27}$ //

Problem 7.5.6

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $f_Y(y)$?
(b) What is $f_{X|Y}(x|y)$?
(c) What is $\mathrm{E}[X|Y=y]$?



a) $f_Y(y) = \int_{-1}^{y} f_{XY}(x,y)\,dx = \int_{-1}^{y} \frac{1}{2}\,dx = \frac{y+1}{2}$

b) $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{2}}{\frac{y+1}{2}} = \frac{1}{y+1}, \quad -1 \le x \le y$

0 o.w

Verilen y için X r.d. -1 ile y arasında düzgün dağılımlı.

c) $E[X|Y=y] = \int_{x=-1}^{y} x\, f_{X|Y}(x)\,dx$

$$= \int_{-1}^{y} \frac{x}{y+1}\,dx$$

$$= \frac{x^2}{2(y+1)}\Big|_{-1}^{y}$$

$$= \frac{y^2-1}{2(y+1)} = \frac{y-1}{2} \qquad -1<y<1$$

Problem 7.6.2

X and Y are jointly Gaussian random variables with $E[X] = E[Y] = 0$ and $\text{Var}[X] = \text{Var}[Y] = 1$. Furthermore, $E[Y|X] = X/2$. Find $f_{X,Y}(x,y)$.

$X, Y \sim N(0,1)$

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}}\, e^{-(y-\tilde{\mu}_Y(x))/2\tilde{\sigma}_Y^2}$$

$$\tilde{\mu}_Y(x) = \mu_Y + \rho_{X,Y}\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$$

$$\tilde{\sigma}_Y(x) = \sigma_Y\sqrt{1-\rho_{XY}^2}$$

$(\mu_Y = 0,\ \mu_X = 0,\ \sigma_Y = \sigma_X = 1)$

$$\tilde{\mu}_Y(x) = \rho_{X,Y}\, x$$

$$E[Y|X] = \rho_{X,Y}\, X \longrightarrow X/2$$

o taman $\rho_{X,Y} = \frac{1}{2}$

$E[Y|X=1) = 1/2$

$E[Y|X=3] = 3/2$

$E[Y|X] = X/2$

$E[E[Y|X]] = E\left[\frac{X}{2}\right] = 0$

$$f_{X,Y}(x,y) = \frac{\exp\left[-\dfrac{x^2 - 2\cdot\frac{1}{2}xy + y^2}{2\left(1-\frac{1}{4}\right)}\right]}{2\pi\sqrt{1-\frac{1}{4}}}$$

$$f_{X,Y}(x,y) = \frac{\exp\left[-\dfrac{x^2-xy+y^2}{3/2}\right]}{2\pi\sqrt{3/4}}$$

Problem 10.1.1

$X_1, \ldots, X_n$ is an iid sequence of exponential random variables, each with expected value 5.

(a) What is $\mathrm{Var}[M_9(X)]$, the variance of the sample mean based on nine trials?

(b) What is $\mathrm{P}[X_1 > 7]$, the probability that one outcome exceeds 7?

(c) Use the central limit theorem to estimate $\mathrm{P}[M_9(X) > 7]$, the probability that the sample mean of nine trials exceeds 7.

$$f_X(x) = \frac{1}{5}e^{-x/5}, \quad x \geq 0 \qquad \lambda = \frac{1}{5}$$

a) $$\mathrm{Var}(M_9(X)) = \mathrm{Var}\left(\frac{X_1 + X_2 + \cdots + X_9}{9}\right)$$

$$= \frac{1}{81} \cdot 9\,\underbrace{\mathrm{Var}(X)}_{\frac{1}{\lambda^2}}$$

$$= \frac{1}{9} \times \frac{1}{25}$$

b) $$P(X_1 > 7) = \int_7^\infty \frac{1}{5}e^{-x/5}\,dx = -e^{-x/5}\Big|_7^\infty = e^{-7/5} \approx 0.25$$

c) Central Limit Theorem

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[\text{dist}]{n\to\infty} N\left(E[X], \frac{\text{Var}(X)}{n}\right)$$

$$M_9(X) \sim N\left(5, \frac{1}{9\times 25}\right)$$

yaklaşık olarak ortalaması 5, ve standart sapması $\frac{1}{15}$ olan bir Gauss r.d. olarak modellenebilir.

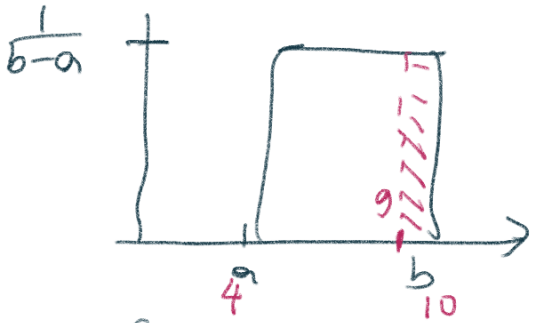
$$P(M_9(X) > 7) \cong P\left(\frac{M_9(X) - 5}{1/15} > \frac{7-5}{1/15}\right)$$

$$\cong Q(30)$$

Problem 10.1.2

$X_1, \ldots, X_n$ are independent uniform random variables with expected value $\mu_X = 7$ and variance $\text{Var}[X] = 3$.

(a) What is the PDF of X_1?
(b) What is $\text{Var}[M_{16}(X)]$, the variance of the sample mean based on 16 trials?
(c) What is $P[X_1 > 9]$, the probability that one outcome exceeds 9?
(d) Would you expect $P[M_{16}(X) > 9]$ to be bigger or smaller than $P[X_1 > 9]$? To check your intuition, use the central limit theorem to estimate $P[M_{16}(X) > 9]$.



$$\mu_X = \frac{a+b}{2} = 7 \rightarrow a+b = 14$$

$$\text{Var}(x) = \frac{(b-a)^2}{12} = 3$$

$$= b - a = 6$$

$$f_X(x) = \begin{cases} \frac{1}{6} & 4 \le x \le 10 \\ 0 & \text{o.w.} \end{cases}$$

$$\boxed{\begin{aligned} b &= 10 \\ a &= 4 \end{aligned}}$$

b) $Var[M_{16}(X)] = Var\left[\frac{X_1 + \cdots + X_{16}}{16}\right]$

$= \frac{Var(X)}{16} = \frac{3}{16}$

c) $P(X_1 > 9) = \frac{1}{6}(10-9) = \frac{1}{6}$

d)

$M_{16}(X) \sim N\left(7, \frac{3}{16}\right)$

$P(M_{16}(X) > 9) = P\left(\frac{M_{16}-7}{3/16} > \frac{9-7}{3/16}\right)$

$\simeq Q\left(\frac{32}{3}\right)$ neredeyse imkansız.

$P(H < h) = P\left(\sqrt{\frac{X^2+Y^2}{2}} < h\right)$

$= P\left(X^2+Y^2 < 2h^2\right)$

$= \iint \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx\,dy$

$0 \le r^2 < 2h^2$

$0 < \theta < 2\pi$

$\leftarrow$ $x^2+y^2 < 2h^2$

$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

Problem 10.2.1

The weight of a randomly chosen Maine black bear has expected value $E[W] = 500$ pounds and standard deviation $\sigma_W = 100$ pounds. Use the Chebyshev inequality to upper bound the probability that the weight of a randomly chosen bear is more than 200 pounds from the expected value of the weight.

std.

$$P(|W - E[W]| > \varepsilon) \leq \frac{Var[W]}{\varepsilon^2}$$

Chebyshev Inequality

$E[W] = 500$

$$P(|W - E[W]| > 200) \leq \frac{100^2}{200^2} \to \frac{1}{4}$$

Problem 10.2.2

For an arbitrary random variable X, use the Chebyshev inequality to show that the probability that X is more than k standard deviations from its expected value $E[X]$ satisfies

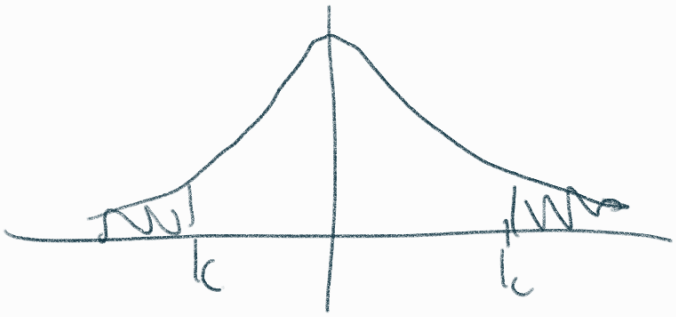
$$P[|X - E[X]| \geq k\sigma] \leq \frac{1}{k^2}.$$

For a Gaussian random variable Y, use the $\Phi(\cdot)$ function to calculate the probability that Y is more than k standard deviations from its expected value $E[Y]$. Compare the result to the upper bound based on the Chebyshev inequality.

Cheb. $P[|X - E[X]| \geq k\sigma] \leq \frac{Var[X]}{k^2\sigma^2} = \frac{1}{k^2}$ $\to \sigma^2$

$Y \sim N(\mu, \sigma)$

$$P[|Y - \mu| \geq k\sigma] = P(Y > \mu + k\sigma) + P(Y < \mu - k\sigma)$$
$$= P\left(\frac{Y-\mu}{\sigma} > \frac{k\sigma}{\sigma}\right) + P\left(\frac{Y-\mu}{\sigma} < \frac{-k\sigma}{\sigma}\right)$$



$$= P(Z > k) + P(Z \le -k)$$
$$= Q(k) + Q(k)$$
$$= 2Q(k)$$

k	Cheb	Gauss	
1	1	$2Q(1)$	$= 2(1-\Phi(1)) \simeq 0.32$
2	$1/4$	$2Q(2)$	$= 2(1-\Phi(2)) = 0.046$
3	$1/9$	$2Q(3)$	$= 2.7 \times 10^{-3}$
4	$1/16$	$2Q(4)$	$= 6.34 \times 10^{-5}$

Problem 10.3.1

Let $X_1, X_2, \ldots$ denote an iid sequence of random variables, each with expected value 75 and standard deviation 15.

(a) How many samples n do we need to guarantee that the sample mean $M_n(X)$ is between 74 and 76 with probability 0.99?

(b) If each X_i has a Gaussian distribution, how many samples n' would we need to guarantee $M_{n'}(X)$ is between 74 and 76 with probability 0.99?

$E[M_n(X)] = 75$

$$P[|M_n(X) - 75| > 1] \le 0.01$$

→ istenen bu.

$n = ?$ $\quad E[X]$

Chebyshev: $P[|M_n(X) - 75| > 1] \le \dfrac{Var(X)}{n \cdot 1^2}$

15^2

$$\frac{Var(X)}{n} \le 0.01$$

$$n > 100\, Var(X) = 15^2 \times 100 = 22500$$

Problem 10.3.2

Let X_A be the indicator random variable for event A with probability $\mathrm{P}[A] = 0.8$. Let $\hat{P}_n(A)$ denote the relative frequency of event A in n independent trials.

(a) Find $\mathrm{E}[X_A]$ and $\mathrm{Var}[X_A]$.

(b) What is $\mathrm{Var}[\hat{P}_n(A)]$?

(c) Use the Chebyshev inequality to find the confidence coefficient $1-\alpha$ such that $\hat{P}_{100}(A)$ is within 0.1 of $\mathrm{P}[A]$. In other words, find α such that

$$\mathrm{P}\left[\left|\hat{P}_{100}(A) - \mathrm{P}[A]\right| \le 0.1\right] \ge 1-\alpha.$$

(d) Use the Chebyshev inequality to find out how many samples n are necessary to have $\hat{P}_n(A)$ within 0.1 of $\mathrm{P}[A]$ with confidence coefficient 0.95. In other words, find n such that

$$\mathrm{P}\left[\left|\hat{P}_n(A) - \mathrm{P}[A]\right| \le 0.1\right] \ge 0.95.$$

$X_A \sim \text{Bernoulli}(0.8)$

$\hat{P}_n(A)$

a) $E[X_A] = 1 \times 0.8 + 0 \times 0.2 = 0.8$

$\mathrm{Var}[X_A] = p(1-p) = 0.8 \times 0.2 = 0.16$

$E[X_A^2] = 1^2 \times 0.8 + 0^2 \times 0.2 = 0.8$

$\mathrm{Var}[X_A] = 0.8 - 0.8^2 = 0.16$

b) $\mathrm{Var}[\hat{P}_n(A)] = \mathrm{Var}\left[\dfrac{X_{A_1} + X_{A_2} + \cdots + X_{A_n}}{n}\right]$

$= \dfrac{\mathrm{Var}[X_A]}{n} = \dfrac{0.16}{n}$

c) Chebyshev $P\left[|X - E[X]| > \varepsilon\right] \le \dfrac{\mathrm{Var}[X]}{n\varepsilon^2}$

$\hat{P}_{100}(A) = \dfrac{\sum_{i=1}^{100} X_{A_i}}{n}$ $\quad P[A] = E[X_A]$

$$P\left[\left|\hat{P}_{100}(A) - P(A)\right| \leq 0.1\right] \geq 1-\alpha$$

$$P\left[\left|\hat{P}_{100}(A) - P(A)\right| \geq 0.1\right] \leq \alpha$$

Chebyshev $\longrightarrow$ $\dfrac{Var(X_A)}{100 \times 0.1^2}$

$$\alpha = \frac{Var(X_A)}{100 \times 0.1^2} \rightarrow 0.16 \quad \Rightarrow \boxed{\alpha = 0.16}$$

d)

$$\alpha = 0.05 = \frac{0.16}{n \times 0.1^2}$$

$$n = \frac{0.16}{0.05 \times 0.01} = 2500 \times 0.16 = 400$$

örnek gerekir

Problem 10.4.1

$X \sim N(\mu, \sigma)$

When X is Gaussian, verify Equation (10.24), which states that the sample mean is within one standard error of the expected value with probability 0.68.

$$M_n(X) = \frac{X_1 + \cdots + X_n}{n}$$

$$Var(M_n(X)) = \frac{Var(X)}{n} = \frac{\sigma^2}{n}$$

$M_n(X)$, $E[X]$'in tarafsız (unbiased) bir kestiricisidir.

$$\left(E[M_n(X)] = E[X]\right)$$

Bu durumda MSE $= Var[M_n(X)] = \dfrac{Var(X)}{n} = e_n$

standard error $= \sqrt{e_n} = \frac{\sigma}{\sqrt{n}}$

$$P\left(\left| M_n(X) - \mu \right| < \frac{\sigma}{\sqrt{n}} \right)$$

$M_n(X)$: kestirim, μ: gerçek

$$P\left(-\frac{\sigma}{\sqrt{n}} < M_n(X) - \mu < \frac{\sigma}{\sqrt{n}} \right)$$

$M_n(X) - \mu \Rightarrow$ beklenen değeri sıfır.

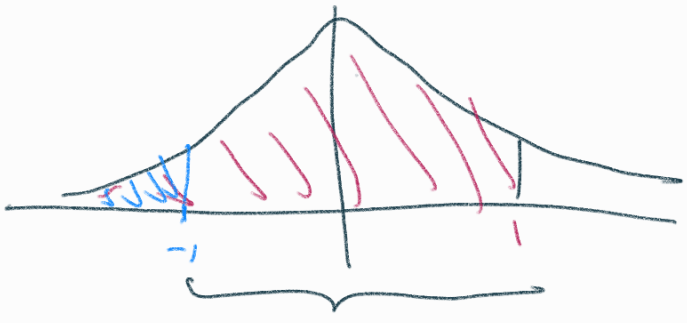
varyans $\frac{\sigma^2}{n}$, std $\frac{\sigma}{\sqrt{n}}$

$$= P\left(-1 < \underbrace{\frac{M_n(X) - \mu}{\sigma/\sqrt{n}}}_{N(0,1)} < 1 \right)$$

$$\Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2 \times 0.84 - 1 = \underline{0.68}$$



$8 = 2\sigma^2 \rightarrow \sigma = 2$

Problem 7.1.6

A random ECE student has height X in inches given by the PDF

$$f_X(x) = \frac{4e^{-(x-70)^2/8} + e^{-(x-65)^2/8}}{5\sqrt{8\pi}} \qquad \sqrt{2\pi}\cdot 2$$

(a) Sketch $f_X(x)$ over the interval $60 \le x \le 75$. (For purposes of sketching, note that $\sqrt{8\pi} \approx 5$.)

(b) Find the probability that a random ECE student is less than 5 feet 8 inches tall. (67 inch)

(c) Use conditional PDFs to explain why $f_X(x)$ might be a reasonable model for ECE students.

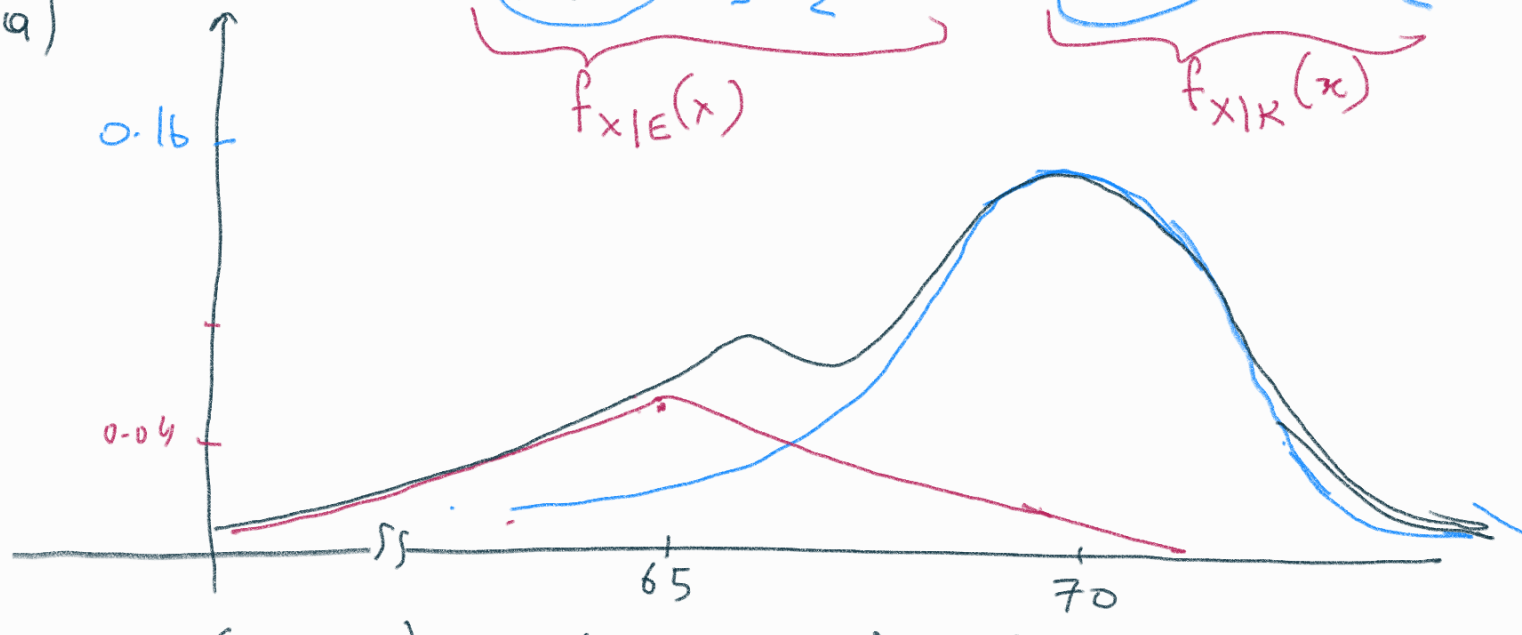
$\sigma = 2$

$-(x-70)^2/8$ $\qquad$ $-(x-65)^2/8$

c) $f_X(x) = 0.8 \dfrac{1}{\sqrt{2\pi}\,2} e^{\;} + 0.2 \dfrac{1}{\sqrt{2\pi}\,2} e^{\;}$ $\rightarrow 0.2$ $\qquad$ $\rightarrow 0.2$

$f_{X|E}(x)$ $\qquad$ $f_{X|K}(x)$

a)



b) $P(X<67) = P(X<67|K)P(K) + P(X<67|E)P(E)$

$P(K) \rightarrow 0.2$ $\qquad$ $P(E) \rightarrow 0.8$

$K: \mu = 65,\ \sigma = 2 \Rightarrow P(X<67|K) = P\left(\frac{X-65}{2} < \frac{67-65}{2}\right)$

$E: \mu = 70,\ \sigma = 2 \Rightarrow P(X<67|E) = P\left(\frac{X-70}{2} < \frac{67-70}{2}\right)$

$\sqrt{}(0,1)$

$= 0.2\,\phi(1) + 0.8\,\phi(-1.5)$

Problem 7.2.4

Y is the exponential (0.2) random variable. Given $A = \{Y < 2\}$, find:
(a) $f_{Y|A}(y)$,
(b) $\mathrm{E}[Y|A]$.

$$f_Y(y) = 0.2\, e^{-0.2x}, \quad x \geq 0$$

a) $$f_{Y|A}(y) = \begin{cases} \dfrac{f_Y(y)}{P(A)} & 0 \leq y \leq 2 \\ 0 & \text{o.w} \end{cases}$$

$$P(A) = P(Y<2) = \int_0^2 0.2\, e^{-0.2x}\, dx$$

$$= F_Y(2) = 1 - e^{-0.2 \times 2} = 1 - e^{-0.4}$$

$$f_{Y|A}(y) = \begin{cases} \dfrac{0.2 e^{-0.2x}}{1 - e^{-0.4}} & 0 \leq y < 2 \\ 0 & \text{o.w} \end{cases}$$

b) $\mathrm{E}[Y|A]$

$$= \int_0^2 y f_{Y|A}(y)\, dy = \int_0^2 \frac{y\, 0.2 e^{-0.2y}}{1 - e^{-0.4}}\, dy$$

$$\left(\begin{array}{ll} 0.2 e^{-0.2y} dy = dv & y = u \\ -e^{-0.2y} = v & dy = du \end{array} \right)$$

$$\frac{1}{1 - e^{-0.4}} \left[-y e^{-0.2y} \Big|_0^2 - \int_0^2 -e^{-0.2y}\, dy \right]$$

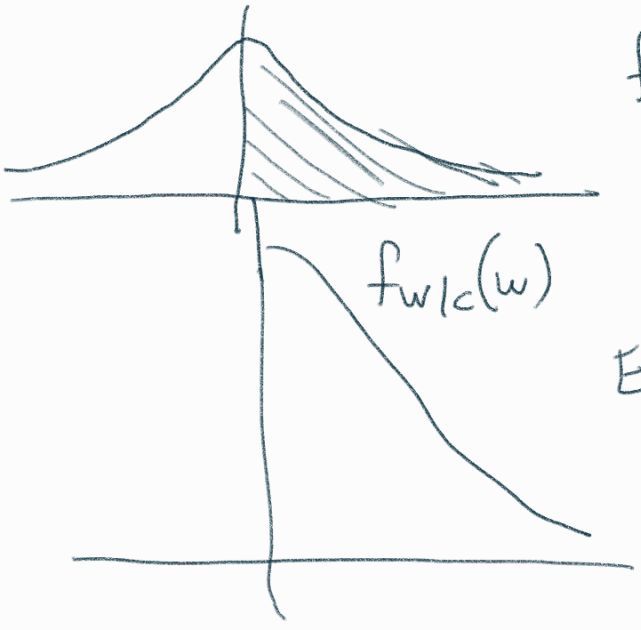
$$\frac{1}{1-e^{-0.4}}\left[-2e^{-0.4} + \left.-5e^{-0.2y}\right|_0^2\right]$$

$$\frac{1}{1-e^{-0.4}}\left(-2e^{-0.4} + 5 - 5e^{-0.4}\right) = \frac{5 - 7e^{-0.4}}{1-e^{-0.4}}$$

Problem 7.2.8

W is the Gaussian $(0,4)$ random variable. Given the event $C = \{W > 0\}$, find the conditional PDF, $f_{W|C}(w)$, the conditional expected value, $\mathrm{E}[W|C]$, and the conditional variance, $\mathrm{Var}[W|C]$.

$$P(C) = P(W>0) = \frac{1}{2}$$



$$f_{W|C}(w) = \begin{cases} \dfrac{f_W(w)}{1/2} & w > 0 \\ 0 & w \leqslant 0 \end{cases}$$

$$E[W|C] = \int_0^\infty 2w \frac{1}{\sqrt{32\pi}} e^{-w^2/32}\, dw$$

$$\frac{w^2}{32} = u$$

$$\frac{w\,dw}{16} = du$$

$$E[W|C] = \int_0^\infty 2\frac{1}{\sqrt{32\pi}} e^{-u}\, 16\, du = \frac{32}{\sqrt{32\pi}} \underbrace{\int_0^\infty e^{-u}\, du}_{1}$$

$$= \frac{32}{\sqrt{32\pi}}$$

$$E[W^2|C] = \int_0^\infty 2w^2 \frac{1}{\sqrt{2\pi}} e^{-w^2/32}\, dw$$

$$= \int_{-\infty}^{\infty} w^2 \underbrace{\frac{1}{\sqrt{2\pi}} e^{-w^2/32}}_{W(0,4)}\, dw = 16$$

$$\mathrm{Var}[W|C] = E[W^2|C] - E[W|C]^2$$

$$= 16 - \left(\frac{32}{\sqrt{32\pi}}\right)^2$$

$$= 16 - \frac{32}{\pi}$$

Problem 7.3.4

N and K have joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{(1-p)^{n-1}p}{n} & n=1,2,\ldots \\ & k=1,\ldots,n, \\ 0 & \text{otherwise.} \end{cases}$$

Let B denote the event that $N \geq 10$.

(a) Find the conditional PMFs $P_{N|B}(n)$ and $P_{N,K|B}(n,k)$. Which should you find first?

(b) Find the conditional expected values $\mathrm{E}[N|B]$, $\mathrm{E}[K|B]$, $\mathrm{E}[N+K|B]$, $\mathrm{Var}[N|B]$, $\mathrm{Var}[K|B]$, $\mathrm{E}[NK|B]$.

$\mathrm{E}[N|B] + \mathrm{E}[K|B]$

$$P(B) = P(N \geq 10) = \sum_{n=10}^{\infty} \underbrace{\sum_{k=1}^{n} \frac{(1-p)^{n-1}p}{n}}$$

$$= \sum_{n=10}^{\infty} (1-p)^{n-1}p$$

$$= (1-p)^9 p \left[1 + (1-p) + (1-p)^2 + \cdots\right]$$

a) $= (1-p)^9 \cdot \frac{1}{1-(1-p)} = (1-p)^9$

$$P_N(n) = \sum_{k=1}^{n} \frac{(1-p)^{n-1} p}{n}$$

$$P_N(n) = (1-p)^{n-1} p, \quad n = 1, \ldots \Rightarrow \text{Geometric}(p)$$

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P_B}, & n = 10, 11, \ldots \\ 0 & n < 10 \end{cases}$$

$$= \begin{cases} \frac{(1-p)^{n-1} p}{(1-p)^9} \to (1-p)^{n-10} p & n = 10, 11, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$P_{N,K|B}(n,k) = \begin{cases} \frac{(1-p)^{n-1} p}{n (1-p)^9} & n = 10, 11, \ldots \\ & k = 1, 2, \ldots, n \\ 0 & \text{o.w.} \end{cases}$$

b) $$E[N|B] = \sum_{n=10}^{\infty} n (1-p)^{n-10} p$$

$n - 10 = m - 1 \Rightarrow m = n - 9$

$$= \sum_{m=1}^{\infty} (m+9) (1-p)^{m-1} p$$

$$= \underbrace{\sum_{m=1}^{\infty} m (1-p)^{m-1} p}_{\frac{1}{p}} + \underbrace{9 \sum_{m=1}^{\infty} (1-p)^{m-1} p}_{9 \cdot 1}$$

$$E[N|B] = \frac{1}{p} + 9$$

$$E[K|B] = ?$$

$$P_K(k) = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1} p}{n}$$

$$E[K|B] = \sum_{k=1}^{n} k \sum_{n=10}^{\infty} P_{K,N|B}(n,k)$$

$$= \sum_{k=1}^{n} k \sum_{n=10}^{\infty} \frac{(1-p)^{n-1} p}{n (1-p)^{9}}$$

Problem 7.4.9

At the One Top Pizza Shop, mushrooms are the only topping. Curiously, a pizza sold before noon has mushrooms with probability $p = 1/3$ while a pizza sold after noon never has mushrooms. Also, a pizza is equally likely to be sold before noon as after noon. On a day in which 100 pizzas are sold, let N equal the number of pizzas sold before noon and let M equal the number of mushroom pizzas sold during the day. What is the joint PMF $P_{M,N}(m,n)$? Are M and N independent? Hint: Find the conditional PMF of M given N.

$N \geq M$ olduğu belli

$$P_{M|N}(m|n) = \frac{P(M=m, N=n)}{P(N=n)} = \frac{\binom{100}{n}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{100-n}}{}$$

$$P_{M|N}(m|n) = \binom{n}{m}\left(\frac{1}{3}\right)^{m}\left(\frac{2}{3}\right)^{n-m}$$

$$P_{M,N}(m,n) = P_{M|N}(m|n)\, P_N(n)$$

$$= \binom{n}{m}\left(\frac{1}{3}\right)^{m}\left(\frac{2}{3}\right)^{n-m}\binom{100}{n}\left(\frac{1}{2}\right)^{100}$$

Problem 7.5.5

Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{(n+1)!} & n=0,1,\ldots; \\ & k=0,1,\ldots,n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PMF $P_N(n)$ and the conditional PMF $P_{K|N}(k|n)$.
(b) Find the conditional expected ~~value~~ $\mathrm{E}[K|N=n]$.
(c) Express the random variable $\mathrm{E}[K|N]$ as a function of N and use the iterated expectation to find $\mathrm{E}[K]$.

a) $$P_N(n) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!} \quad n=0,1,\ldots$$

Poisson(100)

$$P_{K|N}(k|n) = \frac{P_{N,K}(n,k)}{P_N(n)} = \frac{\frac{100^n e^{-100}}{(n+1)!}}{\frac{100^n e^{-100}}{n!}} = \frac{1}{n+1}$$

$k=0,1,\ldots,n$

Düzgün dağılımlı

b) $$E[K|N=n] = \sum_{k=0}^{n} \frac{k}{n+1} = \frac{n}{2}$$

c) $$E[K|N] = \frac{N}{2}$$

$$E[K] = \underbrace{E\left[E[K|N]\right]}_{\text{iterated expectation}} = E\left[\frac{N}{2}\right] = \frac{1}{2}E[N] = \frac{100}{2} = 50$$