2.2.2 A Starburst candy package contains 12 individual candy pieces. Each piece is equally likely to be berry, orange, lemon, or cherry, independent of all other pieces.

(a) What is the probability that a Starburst package has only berry or cherry pieces and zero orange or lemon pieces?

(b) What is the probability that a Starburst package has no cherry pieces?

(c) What is the probability $P[F_1]$ that all twelve pieces of your Starburst are the same flavor?

2.2.6 In a game of poker, you are dealt a five-card hand.

(a) What is the probability $P[R_5]$ that your hand has only red cards?

(b) What is the probability of a "full house" with three-of-a-kind and two-of-a-kind?

a) $\frac{26}{52} \; \frac{25}{51} \; \frac{24}{50} \; \frac{23}{49} \; \frac{22}{48}$

$$\frac{\binom{26}{5}}{\binom{52}{5}} = \frac{\frac{26!}{21!\,\cancel{5!}}}{\frac{52!}{47!\,\cancel{5!}}} =$$

b) 111 22

KK 888

555 AA

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \times 4 \times 12 \times 6 = 3744$$

$$\text{Toplam } \binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2598960$$

Cevap: $P(\text{Fullhouse}) = \frac{3744}{2598960} \cong 0.00144$

$\binom{13}{2} \times 2 \times \binom{4}{3} \times \binom{4}{2}$ bu da aynı.

2.2.7 Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. a) How many different code words are there? b) How many code words have exactly three 0's?

a) 2^5 farklı kod kelimesi olabilir.

b) Sıfırların pozisyonları $\binom{5}{3}$ tane farklı şekilde belirlenebilir.

00011, 00101, 00110, 01001, 01010, 10001, 10010, 11000, 10100, 01100

$\binom{5}{3} = \frac{5 \times 4}{2} = 10$ farklı kod kelimesi

2.2.8 Consider a language containing four letters: A, B, C, D. a) How many three-letter words can you form in this language? b) How many four-letter words can you form if each letter appears only once in each word?

a) 4^3 3 → kelime uzunluğu, 4 → alfabe uzunluğu (sampling with replacement)

b) Sampling without replacement

$4 \times 3 \times 2 \times 1$

$= P(4)_4$

2.2.13 An instant lottery ticket consists of a collection of boxes covered with gray wax. For a subset of the boxes, the gray wax hides a special mark. If a player scratches off the correct number of the marked boxes (and no boxes without the mark), then that ticket is a winner. Design an instant lottery game in which a player scratches five boxes and the probability that a ticket is a winner is approximately 0.01.

kazı kazan

Toplam kutu sayısı N

bunların 5 tanesinde işaret var.

Kazanma olasılığı: $\frac{1}{\binom{N}{5}} = \frac{(N-5)!\,5!}{N!} \simeq 0.01$

$\binom{N}{5}$

$N=6$: olmaz

$N=7$: $\frac{2!\,5!}{7!} = \frac{2}{7 \times 6}$ olmadı

$N=8$: $\frac{3!\,5!}{8!} = \frac{\cancel{6}}{8 \times 7 \times \cancel{6}} = \frac{1}{56}$ olmadı.

$N=9$: $\frac{4!\,5!}{9!} = \frac{\cancel{24}}{\overset{3}{\cancel{9}} \times \cancel{8} \times 7 \times 6} = \frac{1}{126}$ OK.

2.3.1 Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit.

(a) What is the probability of the code word 00111?

(b) What is the probability that a code word contains exactly three ones?

a) $P(00111) = 0.8 \times 0.8 \times 0.2 \times 0.2 \times 0.2$

bitler birbirinden bağımsız

$\cong 0.005$

b) $p = 0.8$

3 ones (yani 2 zeros)

$P(3 \text{ ones}) = \binom{5}{3} p^2 (1-p)^3$

binomial coefficient

iki değerli katsayı

binom katsayısı

c) exactly one 1.

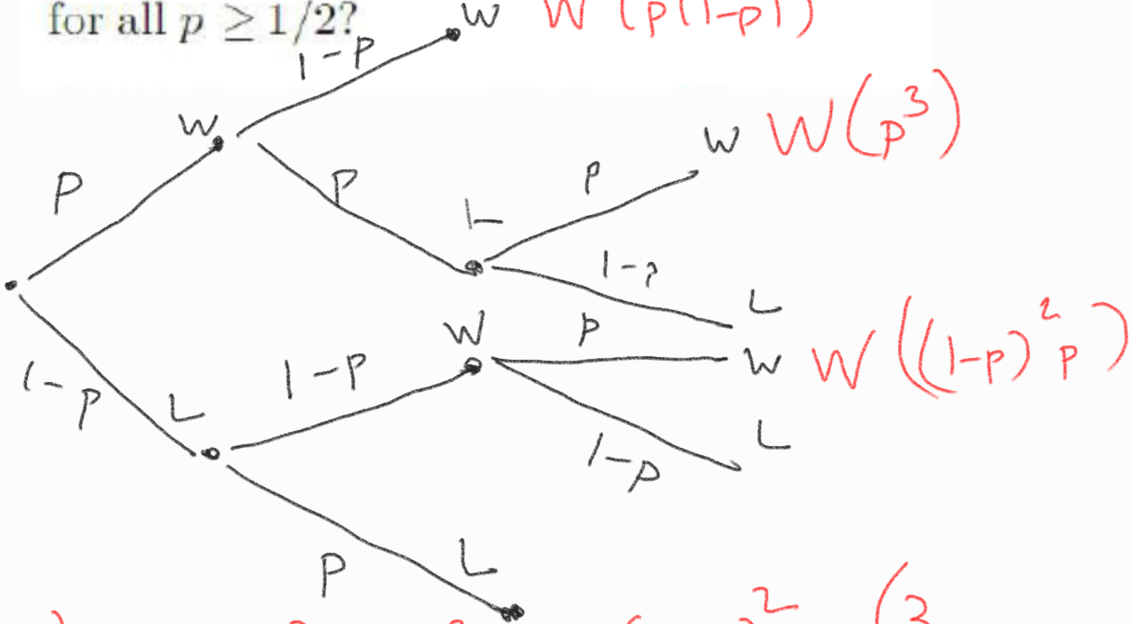
$P(1 \text{ ones}) = \binom{5}{1} p^4 (1-p)^1 = 5 \times 0.8^4 \times 0.2$

d) no 1's

$P(\text{no } 1\text{'s}) = \binom{5}{0} p^5 (1-p)^0 = p^5$

$= (0.8)^5$

2.3.4 In a game between two equal teams, the home team wins with probability $p > 1/2$. In a best of three playoff series, a team with the home advantage has a game at home, followed by a game away, followed by a home game if necessary. The series is over as soon as one team wins two games. a) What is P[H], the probability that the team with the home advantage wins the series? b) Is the home advantage increased by playing a three-game series rather than a one-game playoff? That is, is it true that P[H] $\geq p$ for all $p \geq 1/2$?



$$P(H) = p(1-p) + p^3 + p(1-p)^2 \quad \text{(3-game playoff)}$$

$$P(H') = p \quad \text{(one-game playoff)}$$

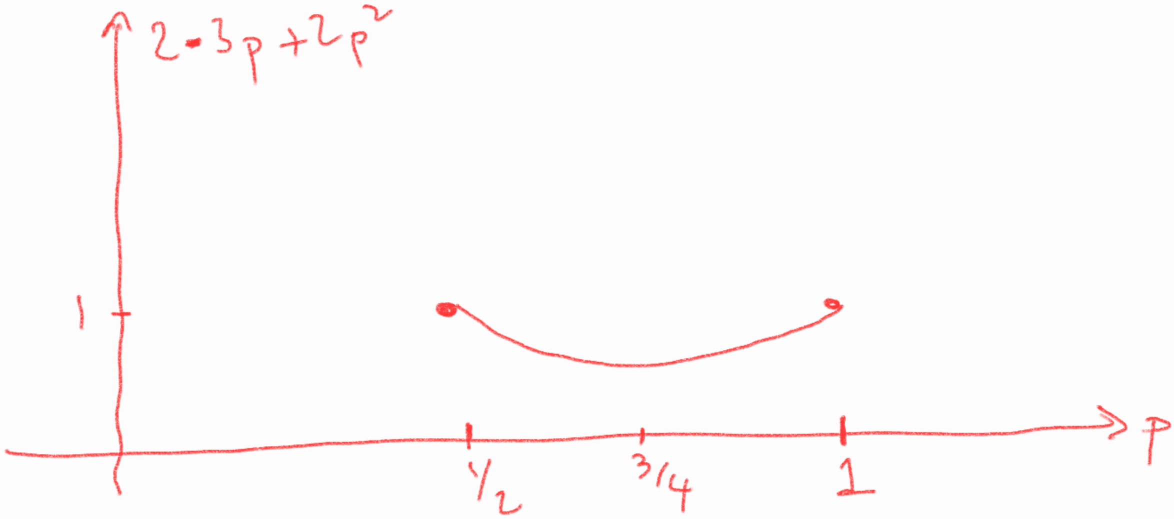
(info: $p > 1/2$)

$$P(H) = p - p^2 + p^3 + p - 2p^2 + p^3$$

$$= 2p - 3p^2 + 2p^3 = p(2 - 3p + 2p^2)$$

$$p\underbrace{(2-3p+2p^2)}\overset{?}{\geq} p$$

$2 - 3/2 + 1/2 = 1$ $\qquad \frac{d}{dp}(2-3p+2p^2) = -3+4p$



Yani $2-3p+2p^2 < 1$

Yani: $p(2-3p+2p^2) < p$

Yani, tek maç olsaydı evsahibi daha avantajlı olurdu.

2.3.5 A collection of field goal kickers are divided into groups 1 and 2. Group i has $3i$ kickers. On any kick, a kicker from group i will kick a field goal with probability $1/(i+1)$, independent of the outcome of any other kicks.

(a) A kicker is selected at random from among all the kickers and attempts one field goal. Let K be the event that a field goal is kicked. Find $P[K]$.

(b) Two kickers are selected at random; K_j is the event that kicker j kicks a field goal. Are K_1 and K_2 independent?

(c) A kicker is selected at random and attempts 10 field goals. Let M be the number of misses. Find $P[M=5]$.

$i = 1, 2$

Group1 : 3 vurucu

Group2: 6 vurucu

Group1: $\frac{1}{2}$ olasılık

Group 2: $\frac{1}{3}$ olasılık

a) $P[K] = P[K|\overset{1/2}{G1}]\,P[\overset{1/3}{G1}] + P[K|\overset{1/3}{G2}]\,P[\overset{2/3}{G2}]$

$P[K] = \frac{1}{6} + 2/9 = \frac{3}{18} + \frac{4}{18} = 7/18$

b) $P[K_1K_2] \stackrel{?}{=} P[K_1]\,P[K_2]$

Yaptığımız 1. seçim 2. vurucunun olasılığını etkiliyor. Bu nedenle K_1, K_2 bağımsız olmayabilir.

c) $P(M=5) = P(M=5 \mid G_1)P(G_1) + P(M=5 \mid G_2)P(G_2)$

$P(M=5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 \frac{1}{3} + \binom{10}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^5 \frac{2}{3}$

2.4.2

$q = 0.1$ bit hata oranı

$P(E) = \binom{5}{5} q^5 (1-q)^0 + \binom{5}{4} q^4 (1-q)^1$

$P(D) = \binom{5}{3} q^3 (1-q)^2 + \binom{5}{2} q^2 (1-q)^3$

$P(E) = (0.1)^5 + 5 \times 0.1^4 \times 0.9$

$= (0.1)^4 (0.1 + 4.5) = 4.6 \times 10^{-4}$

$P(D) \cong 10 \times 0.1^3 \times 0.81 + 10 \times 0.1^2 \times 0.72$

$0.1^2 (0.81 + 7.2) = 8.01 \times 10^{-2}$

$= 0.0801$

$P(C) = \binom{5}{1} q (1-q)^4 + \binom{5}{0} q^0 (1-q)^5$

$= 5 \times 0.1 \times 0.9^4 + 0.9^5$

$= 0.9^4 (0.5 + 0.9) = 1.4 \times 0.9^4 \cong 0.92$

2.4.3 Suppose a 10-digit phone number is transmitted by a cellular phone using four binary symbols for each digit, using the model of binary symbol errors and deletions given in Problem 2.4.2. Let C denote the number of bits sent correctly, D the number of deletions, and E the number of errors. Find $P[C = c, D = d, E = e]$ for all c, d, and e.

Chapter 3: AYRIK RASTGELE DEĞİŞKENLER

Rastgele Değişken
Olasılık Ağırlık Fonksiyonu (PMF)
Dağılımlar
Kümülatif Dağılım Fonksiyonu (CDF)
Beklenen Değer

Problem 3.2.2

The random variable V has PMF

$$P_V(v) = \begin{cases} cv^2 & v = 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.
(b) Find $P[V \in \{u^2 | u = 1, 2, 3, \cdots\}]$.
(c) Find the probability that V is even.
(d) Find $P[V > 2]$.

a) $c(1+4+9+16) = 1$

$c = \frac{1}{30}$

b) $P(V \in (u^2 | u = 1, 2, 3, \ldots)) = P(V \in \{1, 4\})$

$= \frac{1}{30} + \frac{16}{30} = \frac{17}{30}$

c) $P(V \text{ even}) = P(V \in \{2, 4\}) = \frac{4}{30} + \frac{16}{30} = \frac{20}{30}$

d) $P[V > 2] = P[V \in \{3, 4\}] = \frac{9}{30} + \frac{16}{30} = \frac{25}{30}$

Problem 3.2.5

A tablet computer transmits a file over a wi-fi link to an access point. Depending on the size of the file, it is transmitted as N packets where N has PMF

$$P_N(n) = \begin{cases} c/n & n = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) What is the probability that N is odd?

(c) Each packet is received correctly with probability p, and the file is received correctly if all N packets are received correctly. Find $P[C]$, the probability that the file is received correctly.

a) $\frac{c}{1} + \frac{c}{2} + \frac{c}{3} = 1 = c\left(\frac{6+3+2}{6}\right) = 1 \Rightarrow c = \frac{6}{11}$

b) $P[N \text{ tek sayı}] = P[N \in \{1,3\}] = \frac{6}{11}\left(\frac{1}{1} + \frac{1}{3}\right)$

$= 8/11$

c) $P[C] = P[C|N=1]\, P(N=1)$

Toplam olasılık kuralı

$+ P[C|N=2]\, P[N=2]$

$+ P[C|N=3]\, P[N=3]$

$= P \times 6 + p^2 \times 3 + p^3$

Problem 3.2.6

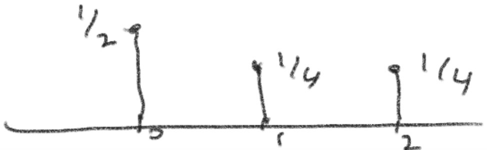
In college basketball, when a player is fouled while not in the act of shooting and the opposing team is "in the penalty," the player is awarded a "1 and 1." In the 1 and 1, the player is awarded one free throw, and if that free throw goes in the player is awarded a second free throw. Find the PMF of Y, the number of points scored in a 1 and 1 given that any free throw goes in with probability p, independent of any other free throw.

$S_Y = \{0, 1, 2\}$

$$P_Y(y) = \begin{cases} 1-p & y = 0 \\ p(1-p) & y = 1 \\ p^2 & y = 2 \\ 0 & \text{o.w} \end{cases}$$

$p = 1/2$ olsa

P.M.F.

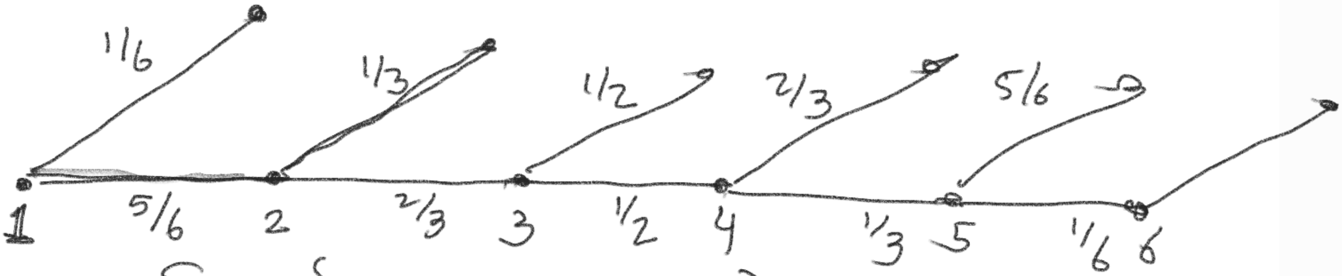


Problem 3.2.7

You roll a 6-sided die repeatedly. Starting with roll $i = 1$, let R_i denote the result of roll i. If $R_i > i$, then you will roll again; otherwise you stop. Let N denote the number of rolls.

(a) What is P[$N > 3$]? $\frac{5}{6} \times \frac{2}{3} \times \frac{1}{2} = \frac{5}{18}$ $\left(\frac{90}{324}\right)$

(b) Find the PMF of N.



$S_N = \{1, 2, 3, 4, 5, 6\}$

$$P_N(n) = \begin{cases} 1/6 & n = 1 \\ 5/18 & n = 2 \\ 5/18 & n = 3 \\ 10/54 & n = 4 \\ \frac{25}{324} & n = 5 \\ 5/324 & n = 6 \\ 0 & \text{o.w} \end{cases}$$

$\frac{5}{6} \times \frac{2}{3} \times \frac{1}{2}$

$\frac{5}{6} \times \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \times \frac{2}{3}$

$\frac{5}{6} \times \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \times \frac{1}{3} \times \frac{5}{6}$

$\frac{5}{6} \times \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \times \frac{1}{3} \times \frac{1}{6}$

36
9
3
4

$$\frac{54 + 90 + 90 + 60 + 25 + 5}{324} = 1$$

Problem 3.2.8

You are manager of a ticket agency that sells concert tickets. You assume that people will call three times in an attempt to buy tickets and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets. Let p be the probability that a caller gets through to your ticket agency. What is the minimum value of p necessary to meet your goal?

$(1-p)^3$ = 3 aramanın da başarısız olması ihtimali

$(1-p)^3 < 0.05$

$1-p < 0.05^{1/3}$

$p > 1-(0.05)^{1/3} = 0.63$

$p > 0.63$ olmalı

Problem 3.2.9

n: number of agents

In the ticket agency of Problem 3.2.8, each telephone ticket agent is available to receive a call with probability 0.2. If all agents are busy when someone calls, the caller hears a busy signal. What is the minimum number of agents that you have to hire to meet your goal of serving 95% of the customers who want tickets?

$p = 1 - 0.8^n$ bir denemenin başarılı olma olasılığı.

0.8^n → available olmama

$(1-p)^3 < 0.05$

$0.8^{3n} < 0.05$

$3n \log 0.8 < \log 0.05$

$n > \dfrac{\log 0.05}{3 \log 0.8} = \dfrac{-1.3}{-3 \times 0.097} = 4.47$

$\boxed{n \geq 5}$ en az 5 kişi olmalı.

Problem 3.2.11

When someone presses SEND on a cellular phone, the phone attempts to set up a call by transmitting a SETUP message to a nearby base station. The phone waits for a response, and if none arrives within 0.5 seconds it tries again. If it doesn't get a response after $n = 6$ tries, the phone stops transmitting messages and generates a busy signal.

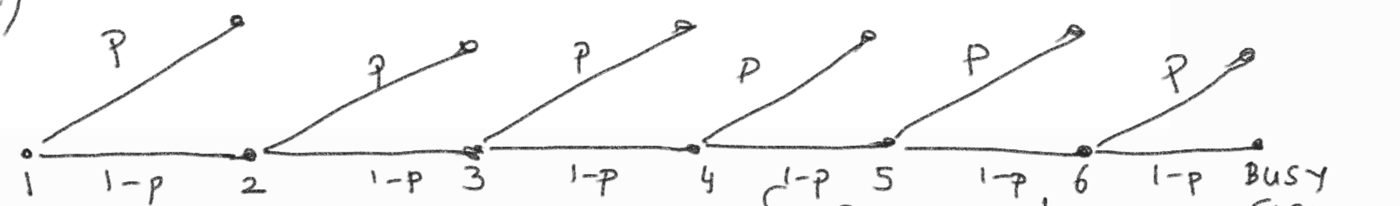
(a) Draw a tree diagram that describes the call setup procedure.

(b) If all transmissions are independent and the probability is p that a SETUP message will get through, what is the PMF of K, the number of messages transmitted in a call attempt?

(c) What is the probability that the phone will generate a busy signal?

(d) As manager of a cellular phone system, you want the probability of a busy signal to be less than 0.02. If $p = 0.9$, what is the minimum value of n necessary to achieve your goal?

a)



$S_K = \{1, 2, 3, 4, 5, 6\}$

b) $P_K(k) = \begin{cases} p & k=1 \\ (1-p)p & k=2 \\ (1-p)^2 p & k=3 \\ (1-p)^3 p & k=4 \\ (1-p)^4 p & k=5 \\ (1-p)^5 p + (1-p)^6 & k=6 \end{cases}$

c) $P(B) = (1-p)^6$

d) $(1-p)^n < 0.02$ $\qquad \frac{1}{10^n} < \frac{1}{50}$

$0.1^n < 0.02$ $\qquad 50 < 10^n$

$50 < 10^n$

$\log_{10} 50 < n$ $\qquad n > 1.7$

$\boxed{n \geqslant 2}$

Problem 3.3.7

When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (~~ACK~~) alındı to the paging system. If the paging system does not receive the ~~ACK~~, it sends the message again.

(a) What is the PMF of N, the number of times the system sends the same message? Geometrik Geom(p)

(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

a) $P_N(n) = P(N=n) = (1-p)^{n-1} p \quad \forall n,\ n = 1,2,3,4,\ldots$

b) $P[N \leqslant 3] \geqslant 0.95$

$1 - P[N > 3] \geqslant 0.95$

$0.05 \geqslant \underbrace{P[N>3]}_{}$

$= P(\text{ilk 3 denemenin başarısız olması})$

$(1-p)^3$

$0.05 \geqslant (1-p)^3$

$(0.05)^{1/3} \geqslant 1-p$

$p \geqslant 1-(0.05)^{1/3} \cong 0.63$

Problem 3.3.8

(a) Starting on day 1, you buy one lottery ticket each day. Each ticket is a winner with probability 0.1. Find the PMF of K, the number of tickets you buy up to and including your fifth winning ticket.

(b) L is the number of flips of a fair coin up to and including the 33rd occurrence of tails. What is the PMF of L?

(c) Starting on day 1, you buy one lottery ticket each day. Each ticket is a winner with probability 0.01. Let M equal the number of tickets you buy up to and including your first winning ticket. What is the PMF of M?

a) $P(K=k) = \underbrace{P\left(\begin{matrix}k-1 \text{ bilet içinde} \\ 4 \text{ ikramiye kazanma}\end{matrix}\right)}_{\text{Binom}} \times \underbrace{P\left(\begin{matrix}k\text{'ıncı bilete} \\ \text{ikramiye çıkması}\end{matrix}\right)}_{\text{Bernoulli}}$

$$\binom{k-1}{4} 0.1^4 \; 0.9^{k-5} \times 0.1$$

$$\boxed{P_K(k) = \binom{k-1}{4} 0.1^5 \; 0.9^{k-5}} \Rightarrow \text{Pascal}(5, 0.1)$$

$k = 5, 6, 7, \ldots$

b) Pascal(33, 0.5)

$$P_K(k) = \binom{l-1}{32} \frac{1}{2}^{33} \frac{1}{2}^{k-33}, \quad k = 33, 34, \ldots$$

c) $P_K(k) = (0.99)^{k-1} \; 0.01 \quad , k = 1, 2, 3, \ldots$ Geom(0.01)

Problem 3.3.10

The number of buses that arrive at a bus stop in T minutes is a Poisson random variable B with expected value $T/5$.

(a) What is the PMF of B, the number of buses that arrive in T minutes?

(b) What is the probability that in a two-minute interval, three buses will arrive?

(c) What is the probability of no buses arriving in a 10-minute interval?

(d) How much time should you allow so that with probability 0.99 at least one bus arrives? $\alpha = T/5$ Poisson$(T/5)$

a) $$P_B(b) = \frac{(T/5)^b \, e^{-(T/5)}}{b!} \qquad b = 0, 1, 2, \ldots$$

c) $\boxed{\begin{matrix} T \longrightarrow T/5 \\ 10 \longrightarrow 2 \end{matrix}}$ otobüs ortalama. Poisson(2)

$$P_B(b) = \frac{2^b \, e^{-2}}{b!}, b = 0, 1, \ldots$$

$$P_B(0) = P(B=0) = \frac{2^0 e^{-2}}{0!} = e^{-2} \simeq 0.135$$

b) $T \longrightarrow T/5$

$2 \longrightarrow 0.4$

Poisson (0.4)

$$P_B(b) = \frac{0.4^b \, e^{-0.4}}{b!}$$

$$P_B(3) = P(B=3) = \frac{0.4^3 \, e^{-0.4}}{3!} = \frac{0.064 \times 0.670}{6} \simeq 0.007$$

$$P(B=1) = \frac{0.4^1 \, e^{-0.4}}{1!} = 0.4 \times 0.67 = 0.268$$

$$P(B>0) = 1 - P(B=0) = 1 - \cancel{\frac{0.4^0}{0!}} e^{-0.4} = 1 - 0.67 = 0.33$$

Problem 3.3.13

In a bag of 64 "holiday season" M&Ms, each M&M is equally likely to be red or green, independent of any other M&M in the bag.

(a) If you randomly grab four M&Ms, what is the probability $P[E]$ that you grab an equal number of red and green M&Ms?

(b) What is the PMF of G, the number of green M&Ms in the bag?

(c) You begin eating randomly chosen M&Ms one by one. Let R equal the number of red M&Ms you eat before you eat your first green M&M. What is the PMF of R?

a) $\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$

b) $P_G(g) = \binom{64}{g}\left(\frac{1}{2}\right)^g\left(\frac{1}{2}\right)^{64-g}, \quad g = 0, 1, \ldots, 64$ (Binom$(64, 1/2)$)

c) $P_R(r) = \begin{cases} \left(\frac{1}{2}\right)^r \frac{1}{2}, & r = 0, 1, \ldots, 63 \\ \left(\frac{1}{2}\right)^{64}, & r = 64 \end{cases}$

Problem 3.3.12

A $(n, \alpha = 1)$ random variable X has PMF

$$P_X(x) = \begin{cases} c(n)/x & x = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

The constant $c(n)$ is set so that $\sum_{x=1}^{n} P_X(x) = 1$. Calculate $c(n)$ for $n = 1, 2, \ldots, 6$.

Problem 3.3.14

A radio station gives a pair of concert tickets to the sixth caller who knows the birthday of the performer. For each person who calls, the probability is 0.75 of knowing the performer's birthday. All calls are independent.

(a) What is the PMF of L, the number of calls necessary to find the winner?

(b) What is the probability of finding the winner on the tenth call?

(c) What is the probability that the station will need nine or more calls to find a winner?

a) Pascal $(6, 0.75)$ $\quad P_L(l) = \binom{l-1}{5} (0.75)^6 \; 0.25^{l-6}, \quad l = 6, 7, \ldots$

b) $P_L(10) = \binom{9}{5} 0.75^6 \; 0.25^3$

c) $\sum_{l=9}^{\infty} P_L(l) = 1 - \sum_{l=6}^{8} P_L(l) = 1 - P_L(6) - P_L(7) - P_L(8)$

Problem 3.3.15

In a packet voice communications system, a source transmits packets containing digitized speech to a receiver. Because transmission errors occasionally occur, an acknowledgment (ACK) or a negative acknowledgment (NAK) is transmitted back to the source to indicate the status of each received packet. When the transmitter gets a NAK, the packet is retransmitted. Voice packets are delay sensitive, and a packet can be transmitted a maximum of d times. If a packet transmission is an independent Bernoulli trial with success probability p, what is the PMF of T, the number of times a packet is transmitted?

$$P_T(t) = \begin{cases} (1-p)^{t-1} p, & t = 1, 2, \ldots, d-1 \\ (1-p)^d + (1-p)^{d-1} p, & t = d \end{cases}$$

Problem 3.3.16

At Newark airport, your jet joins a line as the tenth jet waiting for takeoff. At Newark, takeoffs and landings are synchronized to the minute. In each one-minute interval, an arriving jet lands with probability $p = 2/3$, independent of an arriving jet in any other minute. Such an arriving jet blocks any waiting jet from taking off in that one-minute interval. However, if there is no arrival, then the waiting jet at the head of the line takes off. Each take-off requires exactly one minute.

(a) Let L_1 denote the number of jets that land before the jet at the front of the line takes off. Find the PMF $P_{L_1}(l)$.

10th

(b) Let W denote the number of minutes you wait until your jet takes off. Find P[$W = 10$]. (Note that if no jets land for ten minutes, then one waiting jet will take off each minute and $W = 10$.)

(c) What is the PMF of W?

a) $P_{L_1}(l_1) = \left(\frac{2}{3}\right)^{l_1} \frac{1}{3}, \quad l_1 = 0, 1, \ldots$

b) $\left(\frac{1}{3}\right)^{10} = P[W=10]$

c) Pascal$(10, \frac{1}{3})$

$$P_W(w) = \binom{w-1}{9}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{w-10}, \quad w = 10, 11, \ldots$$

Problem 3.3.17

Suppose each day (starting on day 1) you buy one lottery ticket with probability 1/2; otherwise, you buy no tickets. A ticket is a winner with probability p independent of the outcome of all other tickets. Let N_i be the event that on day i you do *not* buy a ticket. Let W_i be the event that on day i, you buy a winning ticket. Let L_i be the event that on day i you buy a losing ticket.

(a) What are $P[W_{33}]$, $P[L_{87}]$, and $P[N_{99}]$?

(b) Let K be the number of the day on which you buy your first lottery ticket. Find the PMF $P_K(k)$.

(c) Find the PMF of R, the number of losing lottery tickets you have purchased in m days.

(d) Let D be the number of the day on which you buy your jth losing ticket. What is $P_D(d)$? Hint: If you buy your jth losing ticket on day d, how many losers did you have after $d-1$ days?

a) $P[W_{33}] = \frac{1}{2} \times p \longrightarrow$ almak ve kazanmak bağımsız olaylar

$P[L_{87}] = \frac{1}{2} \times (1-p)$

$P[N_{99}] = \frac{1}{2}$

(Not: $\underbrace{P[N_{99}^c]}_{\text{bilet almak}} = \underbrace{P[W_{99}]}_{\text{kazanan bir bilet almak}} + \underbrace{P[L_{99}]}_{\text{kaybeden bir bilet almak}}$

b) Geom$(\frac{1}{2})$

$$P_K(k) = \left(\frac{1}{2}\right)^{k-1}\frac{1}{2}, \quad k = 1, 2, \ldots$$

c) Binom$(m, \frac{1-p}{2})$

$$P_R(r) = \binom{m}{r}\left(\frac{1-p}{2}\right)^{r}\left(1-\frac{1-p}{2}\right)^{m-r}, \quad r = 0, 1, \ldots, m$$

d) Pascal$(j, \frac{1-p}{2})$

$$P_D(d) = \binom{d-1}{j-1}\left(\frac{1-p}{2}\right)^{j}\left(1-\frac{1-p}{2}\right)^{d-j}, \quad d = j, j+1, \ldots$$

Problem 3.3.18

The Sixers and the Celtics play a best out of five playoff series. The series ends as soon as one of the teams has won three games. Assume that either team is equally likely to win any game independently of any other game played. Find $1/2$

(a) The PMF $P_N(n)$ for the total number N of games played in the series;
(b) The PMF $P_W(w)$ for the number W of Celtics wins in the series;
(c) The PMF $P_L(l)$ for the number L of Celtics losses in the series.

a) $P_N(3) = 2\left(\frac{1}{2}\right)^3 = 1/4 \qquad S_N = \{3,4,5\}$

$P_N(4) = 2\binom{4-1}{3-1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 = 2\times 3\times\frac{1}{16} = 6/16$

$P_N(5) = 2\binom{5-1}{3-1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 2\times 6\times\frac{1}{32} = \frac{12}{32}$

$\frac{8}{32} + \frac{12}{32} + \frac{12}{32} = \frac{32}{32}$ ✓

b) $P_W(0)$, $P_W(1)$, $P_W(2)$, $P_W(3) \rightarrow \begin{pmatrix} 3-2 \\ 3-1 \\ 3-0 \end{pmatrix}$ galibiyet (son maç Celtics)

$P_W(0)$: $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$P_W(1) \rightarrow$ 3-1 mağlubiyet durumu. 4. maçta Sixers kazanır.

$\binom{3}{1}\frac{1}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{16}$

$P_W(2) \rightarrow$ 3-2 mağlubiyet (son maç Sixers)

$\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = 6\times\frac{1}{32} = \frac{6}{32}$

$$P_W(w) = \begin{cases} w=0 & \text{w.p. } 1/8 = 4/32 \\ w=1 & \text{w.p. } 3/16 = 6/32 \\ w=2 & \text{w.p. } 6/32 = 6/32 \\ w=3 & \text{w.p. } 16/32 \end{cases}$$

c) $L \longrightarrow S_L = \{0,1,2,3\}$

$P_L(0)$, 3-0 celtics: $\left(\frac{1}{2}\right)^3 = 1/8$

$P_L(1)$, 3-1 celtics: $\binom{3}{2}\left(\frac{1}{2}\right)^3\frac{1}{2} = \frac{3}{16}$

$P_L(2)$, 3-2 celtics: $\binom{4}{2}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{6}{32}$

$P_L(3) \rightarrow \begin{pmatrix} 0-3 \\ 1-3 \\ 2-3 \end{pmatrix}$ Sixers

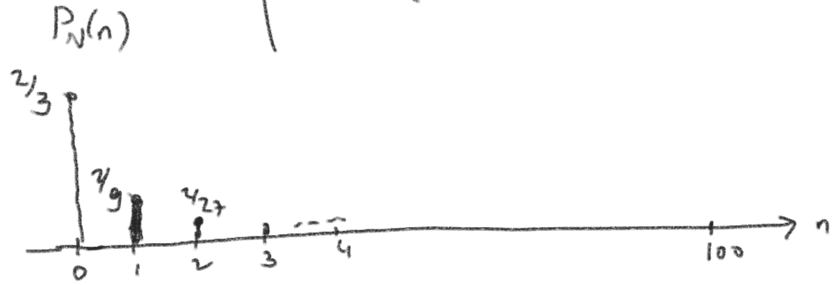
$16/32$

geriye kalan olasılık.

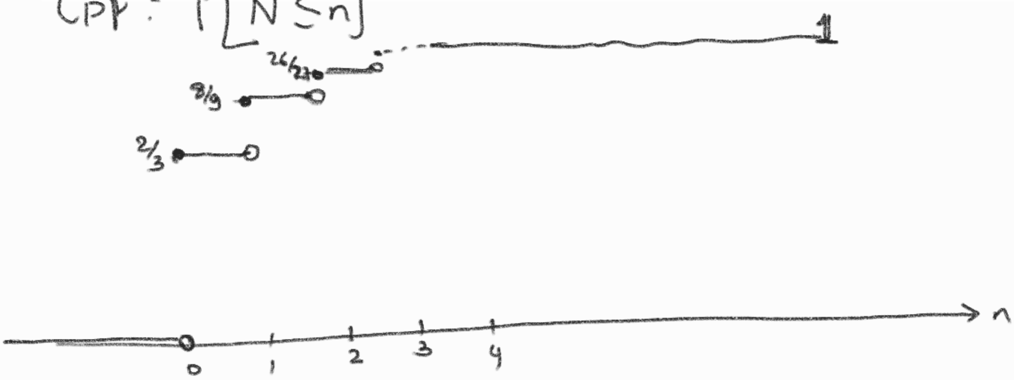
Problem 3.4.4

At the One Top Pizza Shop, a pizza sold has mushrooms with probability $p = 2/3$. On a day in which 100 pizzas are sold, let N equal the number of pizzas sold before the first pizza with mushrooms is sold. What is the PMF of N? What is the CDF of N?

$$P_N(n) = \begin{cases} \left(\frac{1}{3}\right)^n \frac{2}{3}, & n = 0, 1, \ldots, 99 \\ \left(\frac{1}{3}\right)^{100}, & n = 100 \end{cases}$$



CPF: $P[N \leq n]$



Problem 3.5.11

Find $P[K < E[K]]$ when
(a) K is geometric (1/3).
(b) K is binomial $(6, 1/2)$.
(c) K is Poisson (3).
(d) K is discrete uniform $(0, 6)$.

a) $E[K] = 3$

$P[K < 3] = P[K=1] + P[K=2]$

$$\sum_{k=1}^{2} \left(\frac{2}{3}\right)^{k-1} \frac{1}{3}$$

$$= \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} = 5/9$$

b) $E[K] = np$

$= 6 \times \frac{1}{2} = 3$

$P[K < 3] = P[K=0] + P[K=1] + P[K=2]$

$$= \binom{6}{0}\left(\frac{1}{2}\right)^0 \frac{1}{2}^6 + \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4$$

$$= \frac{1}{64} + \frac{6}{64} + \frac{15}{64} = \frac{22}{64}$$

c) Poisson(3)

$E[K] = 3$

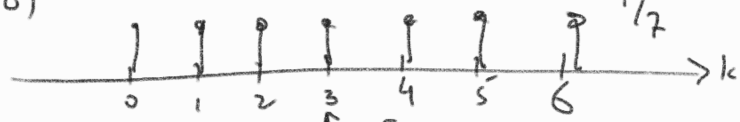
$P[K<3] = P[K=0] + P[K=1] + P[K=2]$

$= e^{-3} + 3e^{-3} + \frac{9e^{-3}}{2} = e^{-3}(8.5) = 0.423$

$$P_K(k) = \frac{3^k e^{-3}}{k!}, \quad k = 0, 1, \dots$$

d) $U(0,6)$

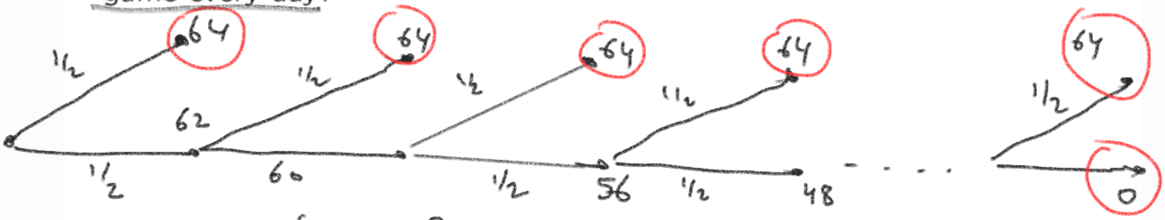
$$\frac{1}{7}\sum_{k=0}^{6} k = 3$$



$$P[K<3] = \underbrace{P[K=0]}_{1/7} + \underbrace{P[K=1]}_{1/7} + \underbrace{P(K=3)}_{1/7} = 3/7$$

Problem 3.5.12

Suppose you go to a casino with exactly \$63. At this casino, the only game is roulette and the only bets allowed are red and green. The payoff for a winning bet is the amount of the bet. In addition, the wheel is fair so that P[red] = P[green] = 1/2. You have the following strategy: First, you bet \$1. If you win the bet, you quit and leave the casino with \$64. If you lose, you then bet \$2. If you win, you quit and go home. If you lose, you bet \$4. In fact, whenever you lose, you double your bet until either you win a bet or you lose all of your money. However, as soon as you win a bet, you quit and go home. Let Y equal the amount of money that you take home. Find $P_Y(y)$ and E[Y]. Would you like to play this game every day?

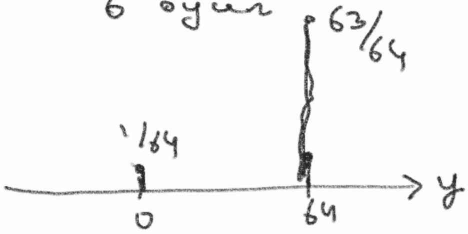


$$S_y = \{0, 64\}$$

$$63 = \underbrace{1 + 2 + 4 + 8 + 16 + 32}_{6 \text{ oyun}}$$

$$P_y(0) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$P_Y(64) = 1 - P_y(0) = \frac{63}{64}$$



P.m.f.

$$P_Y(y) = \begin{cases} 0 & \text{w.p. } 1/64 \\ 64 & \text{w.p. } 63/64 \\ \text{o.w} & \text{w.p. } 0 \end{cases}$$

$$E[Y] = 0 \times \frac{1}{64} + 64 \times \frac{63}{64} = 63\$$$

Problem 3.5.14

You are a contestant on a TV game show; there are four identical-looking suitcases containing \$100, \$200, \$400, and \$800. You start the game by randomly choosing a suitcase. *Among the three unchosen suitcases*, the game show host opens the suitcase that holds the median amount of money. (For example, if the unopened suitcases contain \$100, \$400 and \$800, the host opens the \$400 suitcase.) The host then asks you if you want to keep your suitcase or switch one of the other remaining suitcases. For your analysis, use the following notation for events:

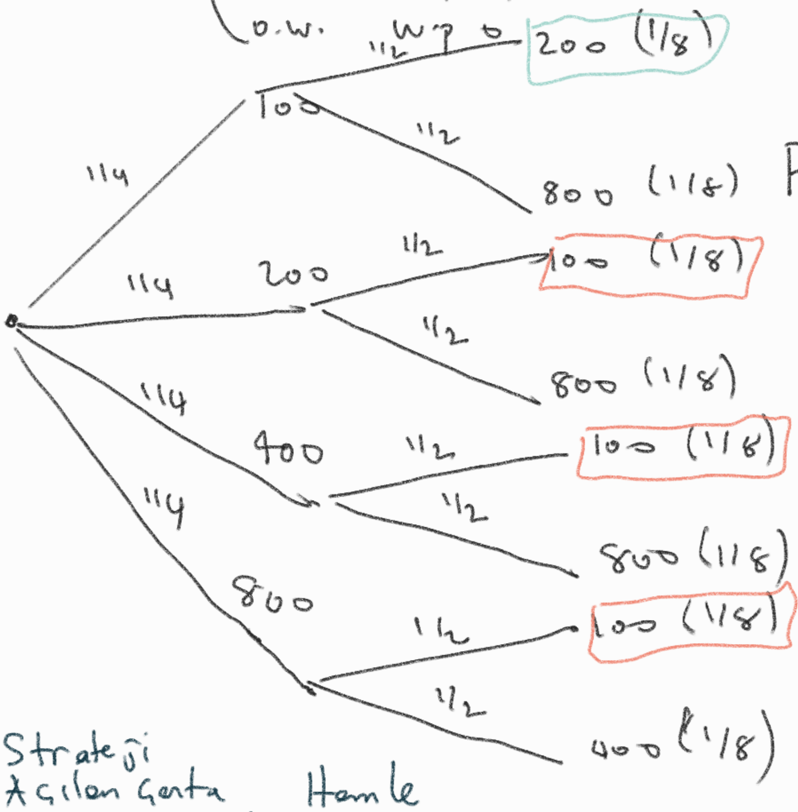
- C_i is the event that you choose a suitcase with i dollars. C_{100} C_{200} C_{400} C_{800}
- O_i denotes the event that the host opens a suitcase with i dollars.
- R is the reward in dollars that you keep.

(a) You refuse the host's offer and open the suitcase you first chose. Find the PMF of R and the expected value $E[R]$.
(b) You always switch and randomly choose one of the two remaining suitcases with equal probability. You receive the R dollars in this chosen suitcase. Sketch a tree diagram for this experiment, and find the PMF and expected value of R.
(c) Can you do better than either always switching or always staying with your original choice? Explain.

a) $P_R(r) = \begin{cases} 100 & \text{w.p } 1/4 \\ 200 & \text{w.p. } 1/4 \\ 400 & \text{w.p } 1/4 \\ 800 & \text{w.p } 1/4 \\ \text{o.w.} & \text{w.p } 0 \end{cases}$

$E[R] = \dfrac{100+200+400+800}{4} = \dfrac{1500}{4} = 375\$$

b) SWITCH



$P_R(r) = \begin{cases} 100 & \text{w.p. } 3/8 \\ 200 & \text{w.p. } 1/8 \\ 400 & \text{w.p. } 1/8 \\ 800 & \text{w.p } 3/8 \\ \text{o.w} & \text{w.p. } 0 \end{cases}$

$E[R] = 100 \times \frac{3}{8} + 200 \times \frac{1}{8} + 400 \times 1/8 + 800 \times 3/8$

$= \dfrac{3300}{8} = 412.5$

DAHA iYi

3300 | 8
32 | 4
1

c)

Strateji Açılan Çanta	Hamle
200	tut
400	değiştir.

Problem 3.5.16

Let binomial random variable X_n denote the number of successes in n Bernoulli trials with success probability p. Prove that $E[X_n] = np$. Hint: Use the fact that $\sum_{x=0}^{n-1} P_{X_{n-1}}(x) = 1$.

p.m.f: $P_{X_n}(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k=0,1,\dots,n$

$$E[X_n] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} k$$

$$= \sum_{\cancel{k=0}}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} k$$

k=1 yazılabilir.

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= pn \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

k-1 yerine k koyalım

$$= np \underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1}}_{\sum_{k=0}^{n-1} P_{X_{n-1}}(k) = 1} = np$$

$$\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= \underbrace{\binom{n-1}{0} p^0 (1-p)^{n}}_{k=1} + \underbrace{\binom{n-1}{1} p^1 (1-p)^{n-2}}_{k=2} + \underbrace{\binom{n-1}{2} p^2 (1-p)^{n-3}}_{k=3}$$

$$+ \dots + \underbrace{\binom{n-1}{n-1} p^{n-1} (1-p)^0}_{k=n}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} = 1$$

Problem 3.5.17

Prove that if X is a nonnegative integer-valued random variable, then

$$\boxed{\mathrm{E}[X] = \sum_{k=0}^{\infty} \mathrm{P}[X > k].}$$

$$E[X] = \sum_{k=0}^{\infty} P_X(k)\, k = \sum_{k=0}^{\infty} P[X=k]\, k$$

$$= P[X=0]\times 0 + P[X=1]\,1 + P(X=2]\,2 + P(X=3)\,3 + \cdots$$

$$= P[X=1] + P[X=2] + P[X=2] + P[X=3] + P[X=3] + P[X=3] + \cdots$$

$$= P[X>0] + P[X>1] + P[X>2] + \cdots$$

$$\boxed{E[X] = \sum_{k=0}^{\infty} P[X>k]}$$

beklenen değeri bulmanın başka bir yolu.

//

Problem 3.5.19

At the gym, a weightlifter can bench press a maximum of 200 kg. For a mass of m kg, $(1 \le m \le 200)$, the maximum number of repetitions she can complete is R, a geometric random variable with expected value $200/m$.

(a) In terms of the mass m, what is the PMF of R?

(b) When she performs one repetition, she lifts the m kg mass a height $h = 4/9.8$ meters and thus does work $w = mgh = 4m$ Joules. For R repetitions, she does $W = 4mR$ Joules of work. What is the expected work $E[W]$ that she will complete?

(c) A friend offers to pay her 1000 dollars if she can perform 1000 Joules of weightlifting work. What mass m in the range $1 \le m \le 200$ should she use to maximize her probability of winning the money?

a) $\boxed{P_{R_m}(r) = \frac{m}{200}\left(1-\frac{m}{200}\right)^{r-1}, \quad r=1,2,\ldots}$

Geom $\left(\frac{m}{200}\right)$

ör: $m = 200$kg olsaydı $P_{R_{200}}(r) = 1 \times 0^{r-1} = 1$, $r=1$

0^0 ?

ör 2: $m = 10$kg $\quad P_{R_{10}}(r) = \frac{1}{20} \times \left(\frac{19}{20}\right)^{r-1} \quad r = 1, 2, \ldots$

$E[R_{10}] = 20.$

b) $P_{W_m}[w] = \begin{cases} \frac{m}{200}\left(1-\frac{m}{200}\right)^0 & w = 4m \\ \frac{m}{200}\left(1-\frac{m}{200}\right)^1 & w = 4m\times 2 \\ \frac{m}{200}\left(1-\frac{m}{200}\right)^2 & w = 4m\times 3 \\ \vdots & \vdots \\ \frac{m}{200}\left(1-\frac{m}{200}\right)^{r-1} & w = 4mr \\ & \vdots \end{cases}$

$E[W_m] = \sum_{r=1}^{\infty} \left(\frac{m}{200}\right)\left(1-\frac{m}{200}\right)^{r-1} 4mr = 4m \underbrace{\sum_{r=1}^{\infty}\left(\frac{m}{200}\right)\left(1-\frac{m}{200}\right)^{r-1} r}_{\frac{200}{m}}$

$$E[X] = \sum_{x \in S_X} P_X(x)\, x$$

$$E[f(X)] = \sum_{x \in S_X} \underbrace{P_X(x)}_{\frac{m}{200}\left(1-\frac{m}{200}\right)^{r-1}} \underbrace{f(x)}_{4mr} \Longrightarrow \text{Bunu uyguladım}$$

$$E[W_m] = 4m \times \frac{200}{m} = \underline{\underline{800 \text{ Joule}}}$$

<u>3.4.2</u> X r.d.

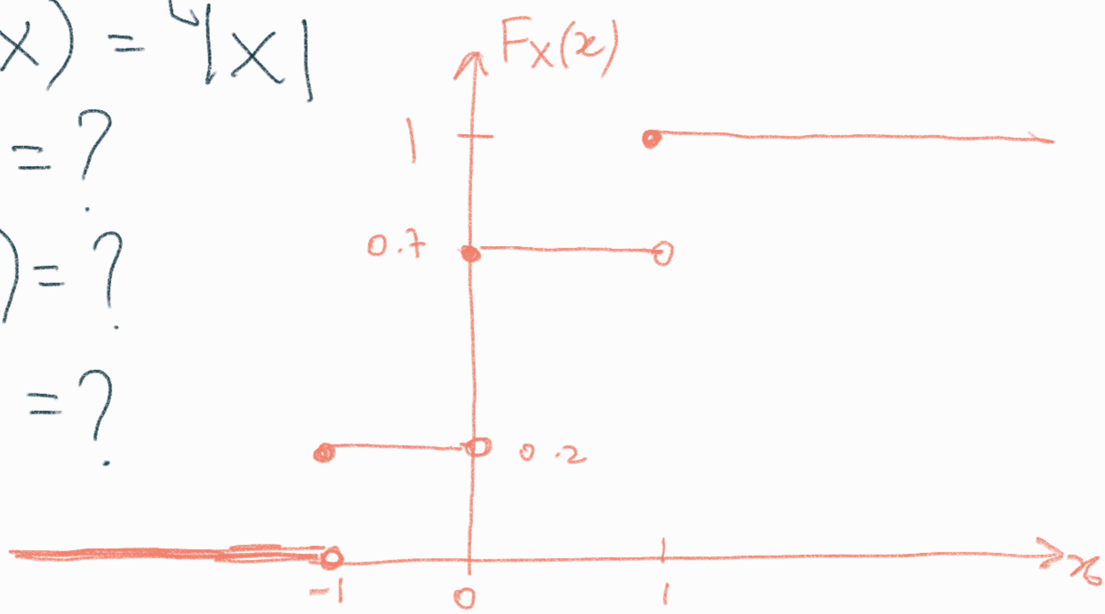
$$F_X(x) = \begin{cases} 0 & x < -1 \\ 0.2 & -1 \leq x < 0 \\ 0.7 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$V = g(X) = |X|$$

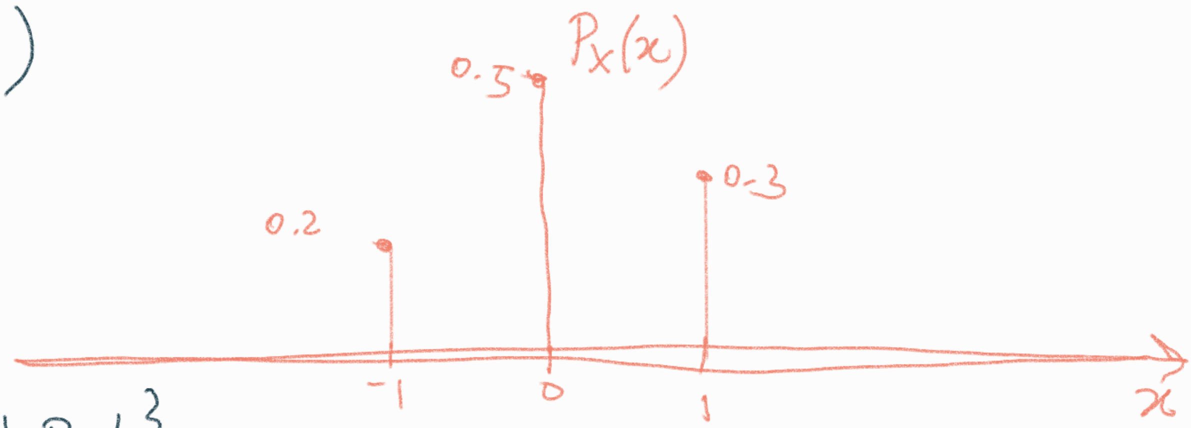
a) $P_V(v) = ?$

b) $F_V(v) = ?$

c) $E[V] = ?$



a)

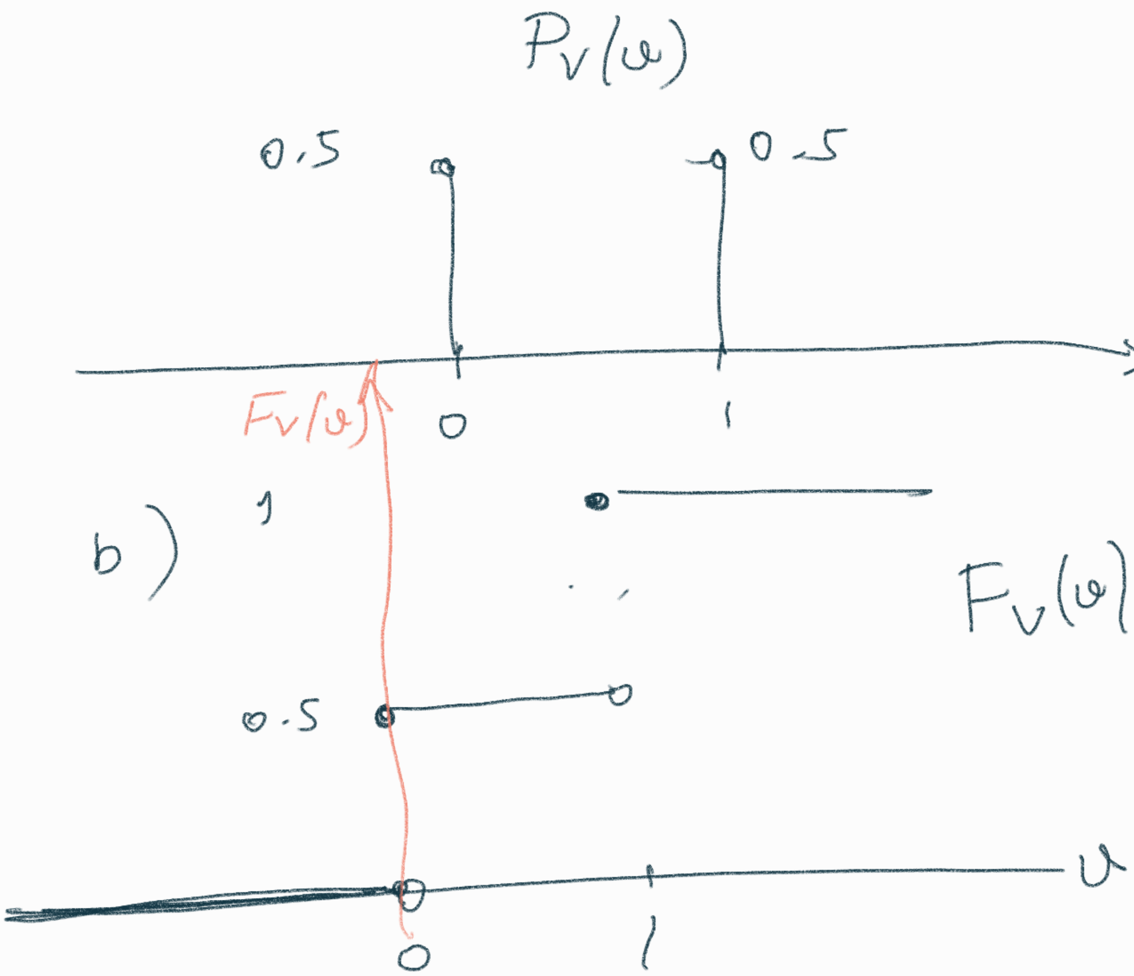


$S_X = \{-1, 0, 1\}$

$S_V = \{0, 1\}$

$P_V(1) = P_X(-1) + P_X(1) = 0.5$

$P_V(0) = P_X(0) = 0.5$



$$F_V(v) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

$= 0.5$

Problem 3.6.4

A source transmits data packets to a receiver over a radio link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is received error free, the receiver sends an acknowledgment (ACK) back to the source. When the receiver gets a packet with errors, a negative acknowledgment (NAK) message is sent back to the source. Each time the source receives a NAK, the packet is retransmitted. We assume that each packet transmission is independently corrupted by errors ~~with pro~~bability q.

(a) Find ~~the PMF of~~ X, the number of times that ~~a packet~~ is transmitted by the source.

(b) ~~Suppose~~ each packet takes 1 millise~~cond to tran~~smit and that the source waits an additional millisecond to receive the acknowledgment message (ACK or NAK) before retransmitting. Let T equal the time required until the packet is successfully received. What is the relationship between ~~T and X? What is the PMF of T?~~

q

Geom(1−q)

a) $P_X(x) = q^{x-1}(1-q)$, $x=1,2,\ldots$

b) $T = 0.002X$

$$P_T(t) = q^{\frac{t}{0.002}-1}(1-q), \quad t=0.002, 0.004, \ldots$$

Not: Eğer NAK olmasaydı.

- Basarsız = 2x 0.001 (Send + Timeout)
- Basarılı = 1x0.001 (Send+ACK) ≈ 0 sn.

$T = (2X-1)\,0.001$

$$\frac{\frac{T}{0.001}+1}{2} = X$$

$$P_T(t) = q^{\frac{\frac{t}{0.001}+1}{2}-1}(1-q), \quad t=0.001, 0.003, 0.005 \ldots$$

Problem 3.7.4

Suppose an NBA basketball player shooting an uncontested 2-point shot will make the basket with probability 0.6. However, if you foul the shooter, the shot will be missed, but two free throws will be awarded. Each free throw is an independent Bernoulli trial with success probability p. Based on the expected number of points the shooter will score, for what values of p may it be desirable to foul the shooter?

No foul : $S \in \{0,2\}$

$P_S(0) = 0.4 \qquad P_S(2) = 0.6$

$E[S] = 0 \times 0.4 + 2 \times 0.6 = 1.2$ sayı

Foul : $S \in \{0,1,2\}$

$P_S(0) = (1-p)^2 \qquad P_S(1) = 2p(1-p)$

$P_S(2) = p^2$

$$E[S] = (1-p)^2 \times 0 + 2p(1-p) \times 1 + p^2 \times 2$$

$$= 2p^2 + 2p(1-p) = \cancel{2p^2} + 2p - \cancel{2p^2}$$

Foul yapmak, $2p < 1.2$ olduğunda

yani $\boxed{p < 0.6}$ olduğunda

Problem 3.7.6

True or False: For any random variable X, $\mathsf{E}[1/X] = 1/\mathsf{E}[X]$.

$$E\left[\frac{1}{X}\right] \overset{?}{=} \frac{1}{E[X]}$$

$$\left(E[f(X)] = f(E[X]) \quad , \quad f(\cdot) \text{ doğrusal ise mümkün (ör. } aX+b) \right)$$

$$\rightarrow E[aX+b] = a\,E[X] + b$$

ancak, $\frac{1}{X}$ doğrusal değil

ör: $$P_X(X) = \begin{cases} 1 & \text{w.p } 0.5 \\ 2 & \text{w.p } 0.5 \\ \text{o.w.} & \text{w.p. } 0 \end{cases}$$

$$E\left[\frac{1}{X}\right] = 0.5 \times \frac{1}{1} + 0.5 \times \frac{1}{2} = 0.75$$

$$\frac{1}{E[X]} = ? \qquad E[X] = 0.5 \times 1 + 0.5 \times 2 = 1.5$$

$$\frac{1}{E[X]} = \frac{1}{1.5} = \frac{2}{3} = 0.67 \neq 0.75$$

FALSE //

Problem 3.7.9

A particular circuit works if all 10 of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each nonworking circuit is worthless and must be thrown away. Each circuit can be built with either ordinary devices or ultrareliable devices. An ordinary device has a failure probability of $q = 0.1$ and costs \$1. An ultrareliable device has a failure probability of $q/2$ but costs \$3. Assuming device failures are independent, should you build your circuit with ordinary devices or ultrareliable devices in order to maximize your expected profit $E[R]$? Keep in mind that your answer will depend on k.

0.05

Ordinary: $W \in \{0, 1\}$

$$P_W(1) = 0.9^{10} \qquad P_W(0) = 1 - 0.9^{10}$$

$$R = W \times k - 10 \times 1$$

$$\boxed{\begin{aligned} E[R] &= P[W=0] \times (-10) + P[W=1] \times (k-10) \\ &= (1-0.9^{10})(-10) + 0.9^{10}(k-10) \end{aligned}}$$

Ultrareliable: $W \in \{0, 1\}$

$$P_W(1) = 0.95^{10} \qquad P_W(0) = 1 - 0.95^{10}$$

$$R = W \times k - 30$$

$$\boxed{\begin{aligned} E[R] &= P[W=0](-30) + P[W=1](k-30) \\ &= (1-0.95^{10})(-30) + 0.95^{10}(k-30) \end{aligned}}$$

Ordinary: $0.9^{10}k - 10$

Ultrareliable: $0.95^{10}k - 30$

$$0.95^{10}k - 30 > 0.9^{10}k - 10$$

$$(0.95^{10} - 0.9^{10})k > 20$$

$$(0.59 - 0.34)k > 20$$

$k > 100$ olmalı

Problem 3.8.4

Let X have the binomial PMF 4 deneme, başarılı sayısı $(p = \frac{1}{2})$

$x = 0,1,2,3,4$ $\qquad P_X(x) = \binom{4}{x}(1/2)^4.$ $\quad \binom{4}{x} \frac{1}{2}^{x} \frac{1}{2}^{4-x}$

(a) Find the standard deviation of X.

(b) What is $P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value?

a) $E[(X - \underbrace{\mu_x}_{E[x]})^2] = E[X^2] - (E[X])^2$

$$E[X] = \left[\underbrace{\binom{4}{0}}_{0} \times 0 + \underbrace{\binom{4}{1}}_{4} \times 1 + \underbrace{\binom{4}{2}}_{12} \times 2 + \underbrace{\binom{4}{3}}_{12} \times 3 + \underbrace{\binom{4}{4}}_{4} \times 4\right] \left(\frac{1}{2}\right)^4$$

$$= 32 \times \left(\frac{1}{2}\right)^4 = 2$$

$$E[X^2] = \left[\underbrace{\binom{4}{0} \times 0^2}_{0} + \underbrace{\binom{4}{1} \times 1^2}_{4} + \underbrace{\binom{4}{2} \times 2^2}_{24} + \underbrace{\binom{4}{3} \times 3^2}_{36} + \underbrace{\binom{4}{4} \times 4^2}_{16}\right] \times \left(\frac{1}{2}\right)^4$$

$$= 80 \times \left(\frac{1}{2}\right)^4 = 5$$

$\sigma_X^2 = 5 - 2^2 = 1$

$\boxed{\sigma_X = 1}$

Not: $\sigma_X^2 = \underset{4}{n}\,\underset{1/2}{p}(1-p) = 1$

b) $P[2-1 \leq X \leq 2+1] = P[1 \leq X \leq 3]$

$= P(X=1) + P(X=2) + P(X=3)$

$\left[\underbrace{\binom{4}{1}}_{4} + \underbrace{\binom{4}{2}}_{6} + \underbrace{\binom{4}{3}}_{4}\right] \times \left(\frac{1}{2}\right)^4 = \frac{14}{16} = \frac{7}{8}$

Problem 3.8.8

In real-time packet data transmission, the time between successfully received packets is called the *interarrival time*, and randomness in packet interarrival times is called *jitter*. Jitter is undesirable. One measure of jitter is the standard deviation of the packet interarrival time. From Problem 3.6.4, calculate the jitter σ_T. How large must the successful transmission probability q be to ensure that the jitter is less than 2 milliseconds?

q : hata ihtimali

1-q : başarı

$X \sim Geom(1-q) \qquad P_X(x) = (1-q)\, q^{x-1}, \; x = 1, 2, \ldots$

$E[X] = \frac{1}{1-q} \qquad E[X^2] =$

$Y = aX$

$\sigma_Y^2 = a^2 \cdot \sigma_X^2$

$\sigma_Y = a\,\sigma_X$

$\boxed{\sigma_X^2 = \frac{q}{(1-q)^2}}$

$T = 2 \times 0.001 \times X$

$\sigma_T^2 = (0.002)^2 \cdot \sigma_X^2$

$= (0.002)^2 \frac{q}{(1-q)^2}$

$\sigma_T = \cancel{0.002} \frac{\sqrt{q}}{1-q} < \cancel{0.002}$

$\sqrt{q} < 1-q$ olmalı

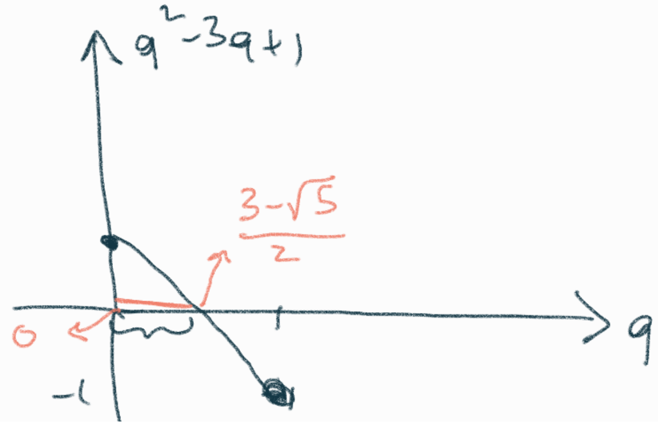
$q < (1-q)^2$

$q < 1 - 2q + q^2$

$\boxed{q^2 - 3q + 1 > 0}$

$$q = \frac{3 \mp \sqrt{5}}{2}$$

türev $2q-3$



$$\boxed{0 < q < \frac{3-\sqrt{5}}{2} \text{ olmalı}}$$

Problem 3.8.9

Random variable K has a Poisson (α) distribution. Derive the properties $E[K] = \text{Var}[K] = \alpha$. Hint: $E[K^2] = E[K(K-1)] + E[K]$.

$K^2 = K(K-1) + K$

$$P_K(k) = \frac{\alpha^k e^{-\alpha}}{k!}, \quad k = 0, 1, \ldots$$

$$E[K] = \sum_{\cancel{k=0}}^{\infty} k \frac{\alpha^k e^{-\alpha}}{k!} = \sum_{k=1}^{\infty} \frac{\alpha^k e^{-\alpha}}{(k-1)!} = \alpha \sum_{k=1}^{\infty} \frac{\alpha^{k-1} e^{-\alpha}}{(k-1)!}$$

$k=1$ yazabiliriz

$k-1=u$

$$= \alpha \underbrace{\sum_{u=0}^{\infty} \frac{\alpha^u e^{-\alpha}}{u!}}_{1} = \alpha$$

$$\boxed{E[K] = \alpha}$$

$$E[K^2] = \underbrace{E[K(K-1)]}_{} + \underbrace{E[K]}_{\alpha}$$

$$E[K(K-1)] = \sum_{\cancel{k=0}}^{\infty} k(k-1) \frac{\alpha^k e^{-\alpha}}{k!} = \sum_{k=2}^{\infty} \frac{\alpha^2 \alpha^{k-2} e^{-\alpha}}{(k-2)!}$$

$k=2$ yazılabilir

$u=k-2$

$$= \alpha^2 \underbrace{\sum_{u=0}^{\infty} \frac{\alpha^u e^{-\alpha}}{u!}}_{1} = \alpha^2$$

$E[K(K-1)] = \alpha^2$

$$E[K^2] = \alpha^2 + \alpha$$

$$\sigma_K^2 = E[K^2] - (E[K])^2 = \cancel{\alpha^2} + \alpha - \cancel{\alpha^2}$$

$$\sigma_K^2 = \alpha$$

Problem 3.8.1

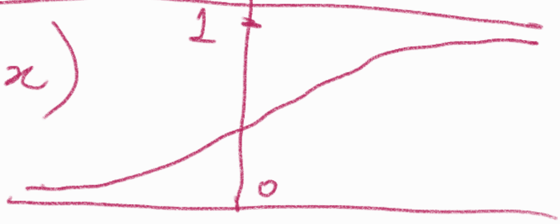
In an experiment to monitor two packets, the PMF of N, the number of video packets, is

n	0	1	2
$P_N(n)$	0.2	0.7	0.1

Find $\mathrm{E}[N]$, $\mathrm{E}[N^2]$, $\mathrm{Var}[N]$, and σ_N.

CHAPTER 4: CONTINUOUS RANDOM VARIABLES

4.2 CDF $\quad F_X(x) = P(X \leq x)$



4.3 PDF $\quad f_X(x)$

Prob. Density Func.

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}$$

$$\int_a^b f_X(x)\,dx = P[a \leq X \leq b]$$
$$= P[a < X \leq b]$$
$$= P[a \leq X < b]$$
$$= P[a < X < b]$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$F_X(x) = \int_{u=-\infty}^{x} f_X(u)\,du = P[X \leq x]$$

4.4 Expected Value (Beklenen Değer)

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \mu_X$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$E[g(X)] = \int_{x=-\infty}^{\infty} g(x)\, f_X(x)\,dx$$

$$Var(X) = E[(X-\mu_X)^2] = E[X^2] - (E[X])^2$$

Problem 4.2.4

The CDF of random variable W is

$$F_W(w) = \begin{cases} 0 & w < -5, \\ \frac{w+5}{8} & -5 \le w < -3, \\ \frac{1}{4} & -3 \le w < 3, \\ \frac{1}{4} + \frac{3(w-3)}{8} & 3 \le w < 5, \\ 1 & w \ge 5. \end{cases}$$

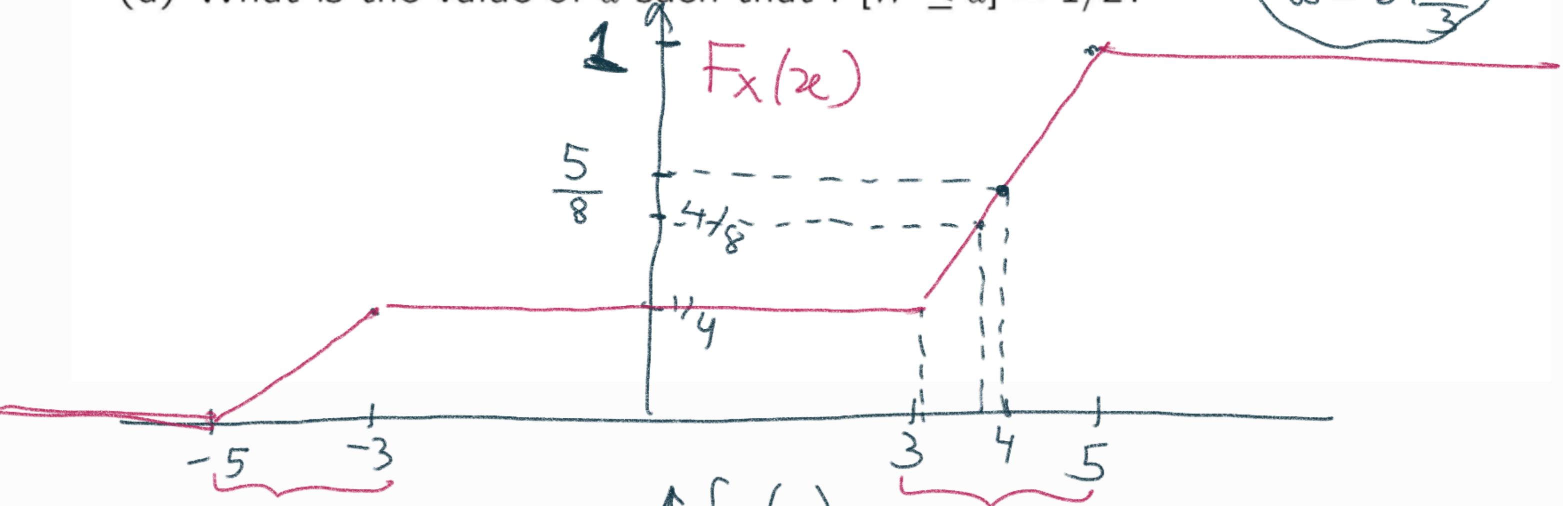
$$\frac{1}{4} + \frac{3(w-3)}{8} = \frac{1}{2}$$

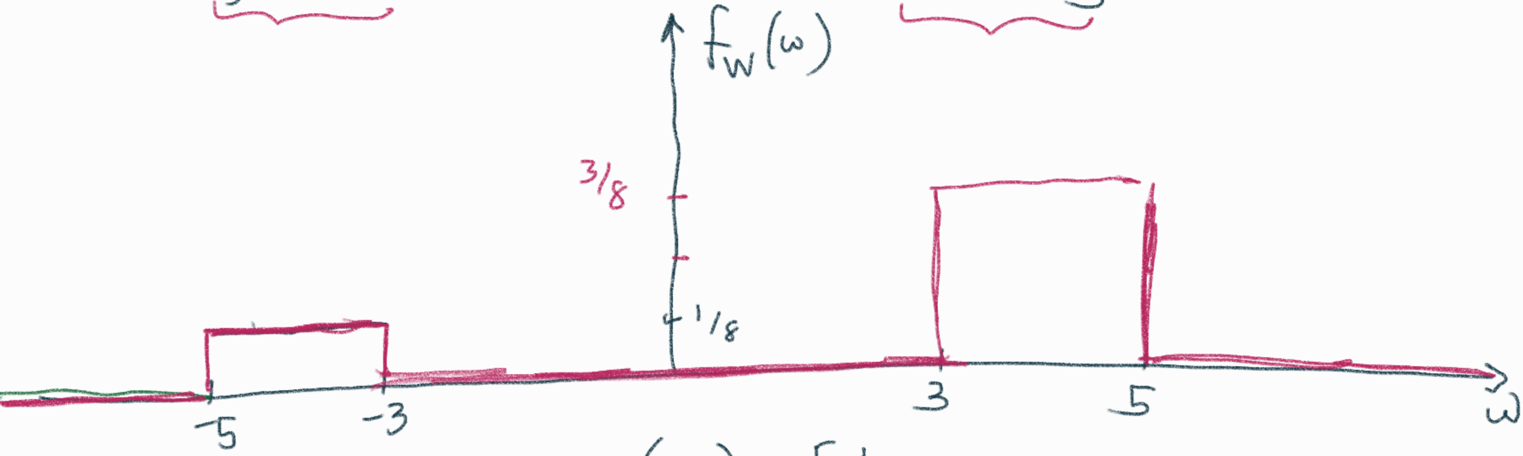
$$\frac{3(w-3)}{8} = \frac{1}{4}$$

$$w - 3 \qquad \frac{2}{3}$$

$$w = 3 + \frac{2}{3}$$

(a) What is P[$W \le 4$]?
(b) What is P[$-2 < W \le 2$]?
(c) What is P[$W > 0$]?
(d) What is the value of a such that P[$W \le a$] $= 1/2$?





a) $P(W \le 4) = F_W(4) = 5/8$

$$\underbrace{\int_{-5}^{-3} f_X(x)\,dx}_{2/8} + \underbrace{\int_{3}^{4} f_X(x)\,dx}_{3/8} = 5/8$$

b) $P(-2 \le W \le 2) = \int_{-2}^{2} f_X(x)\,dx = \emptyset$

$F_W(2) - F_W(-2) = \emptyset$

c) $P[W>0] = \int_0^{\infty} f_X(x)\,dx = \int_3^5 f_X(x)\,dx = 3/4$

$= 1 - P[W \leq 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$

d) $P[W \leq a] = 1/2$

$\downarrow$

$F_W(a) = \frac{1}{2} \Rightarrow a = 3 + \frac{2}{3} = \frac{11}{3}$

a: medyan.

Problem 4.3.3

Find the PDF $f_U(u)$ of the random variable U in Problem 4.2.4.

Yukarıda çizdik.

$$f_W(w) = \begin{cases} 0 & w \leq -5 \\ 1/8 & -5 < w \leq -3 \\ 0 & -3 < w \leq 3 \\ 3/8 & 3 < w \leq 5 \\ 0 & 5 \leq w \end{cases}$$

Problem 4.3.4

For a constant parameter $a > 0$, a Rayleigh random variable X has PDF

$$f_X(x) = \begin{cases} a^2 x e^{-a^2x^2/2} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the CDF of X?

(Not : $X \sim$ Gauss, $Y \sim$ Gauss, $|X + Y| \sim$ Rayleigh)

$$F_X(x) = \int_{-\infty}^{x} f_X(r)\,dr = \int_0^x a^2 r e^{-a^2r^2/2}\,dr$$

$$= -e^{-a^2r^2/2}\Big|_0^x$$

$$\left[\frac{d}{dr}\left(-e^{-a^2r^2/2}\right) = a^2 r e^{-a^2r^2/2}\right]$$

$$\boxed{F_X(x) = 1 - e^{-a^2x^2/2}, \quad x > 0}$$

Problem 4.3.5

Random variable X has a PDF of the form $f_X(x) = \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)$, where

$$f_1(x) = \begin{cases} c_1 & 0 \le x \le 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_2(x) = \begin{cases} c_2 e^{-x} & x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

What conditions must c_1 and c_2 satisfy so that $f_X(x)$ is a valid PDF?

Geçerli p.d.f. $\longrightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\longrightarrow f_X(x) \geq 0 \quad \forall x$

$$f_X(x) = \begin{cases} \frac{c_1}{2} + \frac{c_2}{2}e^{-x} & 0 \le x < 2 \\ c_2 e^{-x} & 2 \le x \\ 0 & \text{o.w} \end{cases}$$

1. $\frac{c_1}{2} + \frac{c_2}{2}e^{-x} \geq 0 \quad 0 \le x < 2$

2. $c_2 e^{-x} \geq 0, \; x \geq 2 \longrightarrow c_2 \geq 0$

3. $\int_0^2 \frac{c_1}{2} + \frac{c_2}{2}e^{-x}\,dx + \int_2^{\infty} c_2 e^{-x}\,dx = 1$

$$\left.\frac{c_1 x}{2} - \frac{c_2}{2}e^{-x}\right|_0^2 + \left.-c_2 e^{-x}\right|_2^{\infty} = 1$$

$$\boxed{c_1 - \frac{c_2}{2}e^{-2} + \frac{c_2}{2} + c_2 e^{-2} = 1}$$

$$\boxed{c_1 + \frac{c_2}{2} + \frac{c_2}{2} e^{-2} = 1}$$

Problem 4.4.1

Random variable X has PDF

uniform r.v.

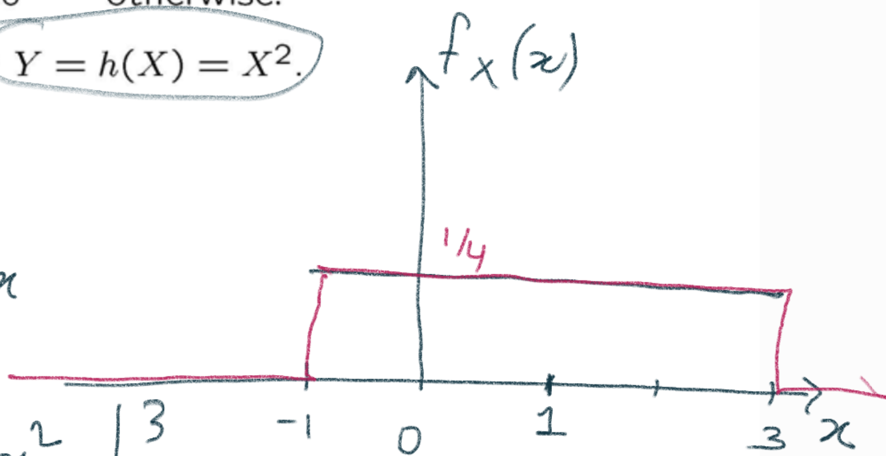
$$f_X(x) = \begin{cases} 1/4 & -1 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Define the random variable Y by $Y = h(X) = X^2$.

(a) Find E$[X]$ and Var$[X]$.

(b) Find $h(\text{E}[X])$ and E$[h(X)]$.

(c) Find E$[Y]$ and Var$[Y]$.



a) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= \int_{-1}^{3} x \frac{1}{4} dx = \frac{x^2}{8}\Big|_{-1}^{3} = \frac{9-1}{8} = 1$$

$$Var(x) = E[X^2] - (E[X])^2 = E[X^2] - 1$$

$$E[X^2] = \int_{x=-1}^{3} x^2 \frac{1}{4} dx = \frac{x^3}{12}\Big|_{x=-1}^{3} = \frac{27+1}{12} = \frac{28}{12}$$

$$E[g(X)] = \int_{x=-\infty}^{\infty} g(x) f_X(x) dx$$

$$Var(x) = \frac{28}{12} - 1 = \frac{16}{12} = \frac{4}{3} = \sigma_X^2$$

b) $h(E[X]) = (E[X])^2 = 1$

$E[h(X)] = E[X^2] = 28/12$

Eşit değil
Çünkü fonksiyon
doğrusal
değil

c) $E[Y] = E[X^2] = 28/12 = \frac{7}{3}$

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2$$

$$= E[X^4] - \left(\frac{28}{12}\right)^2$$

$$E[X^4] = \int_{x=-1}^{3} x^4 \frac{1}{4} dx = \frac{x^5}{20}\Big|_{-1}^{3} = \frac{3^5 - (-1)^5}{20}$$

$$= \frac{243+1}{20} = \frac{244}{20} = 12.2$$

$$\mathrm{Var}(Y) = 12.2 - \frac{49}{9} = 6.76$$

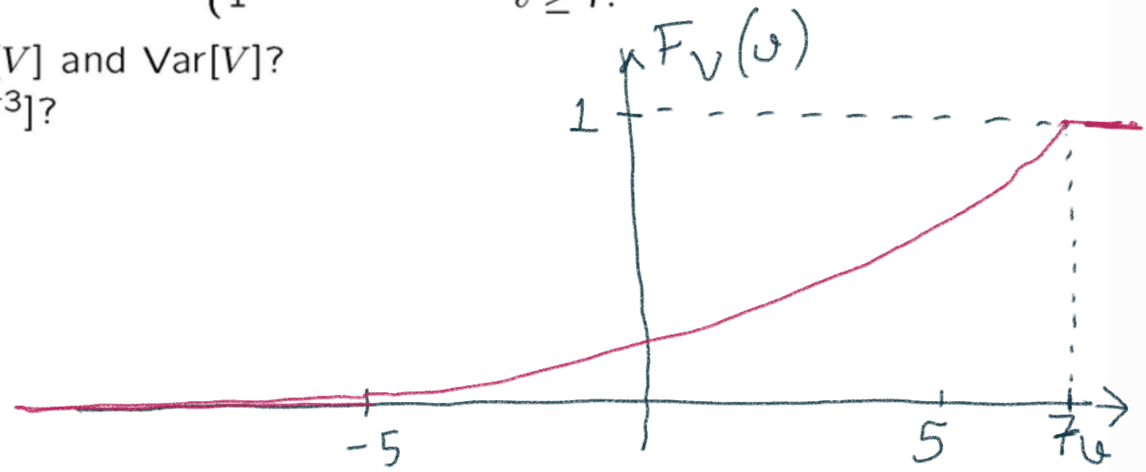
Problem 4.4.6

The cumulative distribution function of random variable V is

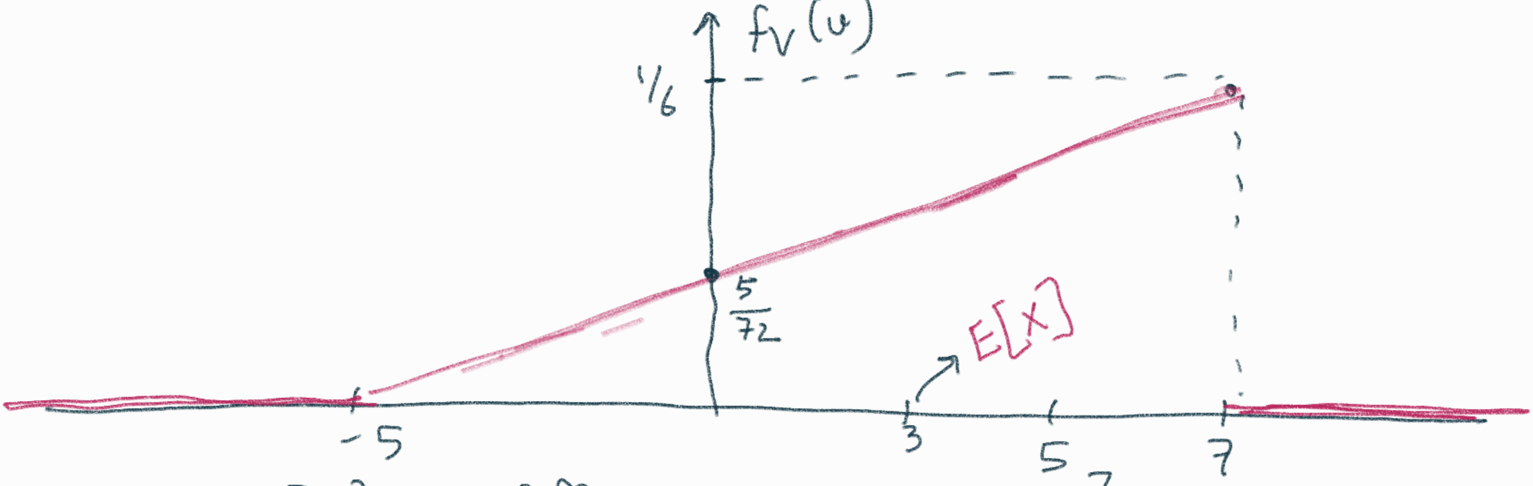
$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144 & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What are E[V] and Var[V]?

(b) What is E[V^3]?



$$f_V(v) = \begin{cases} 0 & v < -5 \\ \frac{v+5}{72} & -5 < v < 7 \\ 0 & v \geq 7 \end{cases}$$



a) $E[V] = \int_{-\infty}^{\infty} v \, f_V(v) \, dv = \int_{-5}^{7} v \frac{(v+5)}{72} \, dv$

$= \left. \frac{v^3}{3 \times 72} + \frac{5v^2}{144} \right|_{-5}^{7} = 2.17 + 0.83 = 3$

b) $E[V^2] = \int_{-5}^{7} \frac{v^2(v+5)}{72} \, dv = \frac{v^4}{4 \times 72} + \frac{5v^3}{3 \times 72}$

$= 6.17 + 10.83$

$\cong 17$

$Var[V] = E[V^2] - (E[V])^2$

$= 17 - 9 = 8$

$Std[V] = \sqrt{8}$

c) $E[V^3] = \int_{-5}^{7} \frac{v^3(v+5)}{72} \, dv = \left. \frac{v^5}{360} + \frac{5v^4}{288} \right|_{-5}^{7}$

$= 55.37 + 30.83$

$= 86.20$

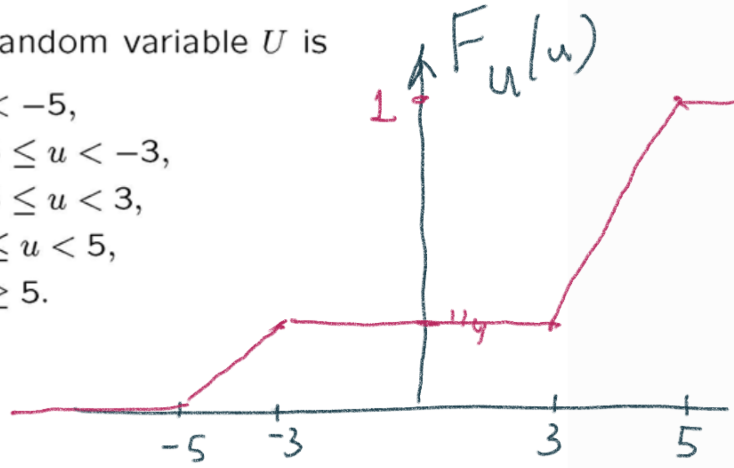
Problem 4.4.7

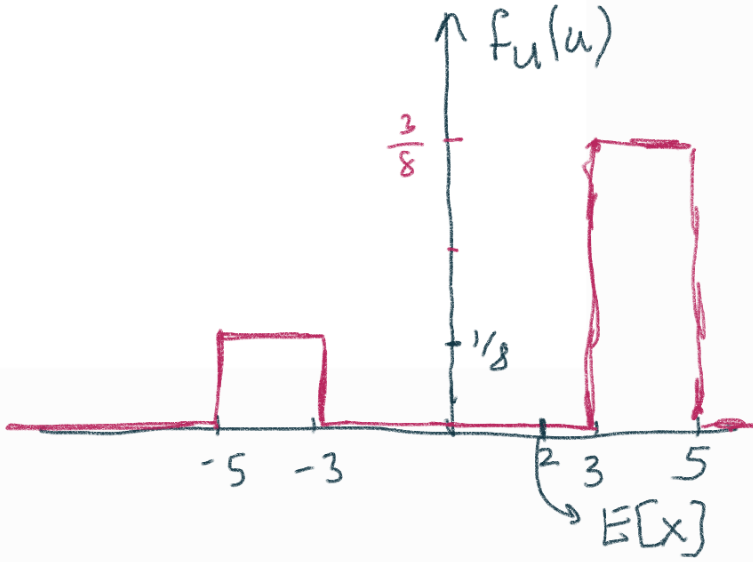
The cumulative distribution function of random variable U is

$$F_U(u) = \begin{cases} 0 & u < -5, \\ \frac{u+5}{8} & -5 \le u < -3, \\ \frac{1}{4} & -3 \le u < 3, \\ \frac{3u-7}{8} & 3 \le u < 5, \\ 1 & u \ge 5. \end{cases}$$

(a) What are E[U] and Var[U]?

(b) What is E[2^U]?





a) $$E[U] = \int_{-5}^{-3} u \frac{1}{8} du + \int_{3}^{5} u \frac{3}{8} du$$

$$\frac{u^2}{16}\Big|_{-5}^{-3} + \frac{3u^2}{16}\Big|_{3}^{5}$$

$$= \frac{9-25}{16} + \frac{3(25-9)}{16}$$

$$= 2 \;//$$

$$E[U^2] = \int_{-5}^{-3} \frac{u^2}{8} du + \int_{3}^{5} \frac{3u^2}{8} du = \frac{u^3}{24}\Big|_{-5}^{-3} + \frac{u^3}{8}\Big|_{3}^{5}$$

$$= \frac{-27+125}{24} + \frac{125-27}{8}$$

$$= 8.16$$

$$Var(U) = 8.16 - 2^2 = 4.16 \;// \qquad std(U) \cong 2.1 \;//$$

b) $$E[2^U] = \int_{-5}^{-3} \frac{2^u}{8} du + \int_{3}^{5} \frac{3 \cdot 2^u}{8} du$$

$$= \frac{1}{8\ln 2} 2^u \Big|_{-5}^{-3} + \frac{3}{8 \cdot \ln 2} \cdot 2^u \Big|_{3}^{5}$$

$$\int b^x\,dx = \int \left[\left(e^{\ln b}\right)^x\right] dx = \int e^{(\ln b)x}\,dx$$

$$u = (\ln b)x \to du = \ln b \; dx$$

$$= \int e^u \frac{du}{\ln b} = \frac{1}{\ln b}\int e^u\,du$$

$$\frac{2^{-3} - 2^{-5}}{8\ln 2} + \frac{3}{8\ln 2}\left(2^5 - 2^3\right) = \frac{9}{\ln 2}$$

Problem 4.4.8

X is a Pareto (α, μ) random variable, as defined in Appendix A. What is the largest value of n for which the nth moment $E[X^n]$ exists? For all feasible values of n, find $E[X^n]$.

Pareto (α, μ), $\alpha > 0$, $\mu > 0$

$$f_X(x) = \begin{cases} \left(\frac{\alpha}{\mu}\right)\left(\frac{x}{\mu}\right)^{-(\alpha+1)} & x \geq \mu \\ 0 & \text{o.w} \end{cases}$$

$$E[X^n] = \int_\mu^\infty \left(\frac{\alpha}{\mu}\right) x^n \left(\frac{x}{\mu}\right)^{-(\alpha+1)} dx$$

$$= \mu^n \int_\mu^\infty \left(\frac{\alpha}{\mu}\right)\left(\frac{x}{\mu}\right)^n \left(\frac{x}{\mu}\right)^{-(\alpha+1)} dx$$

$$= \mu^n \int_\mu^\infty \left(\frac{\alpha}{\mu}\right)\left(\frac{x}{\mu}\right)^{-(\alpha+1-n)} dx$$

$$= \frac{\mu^n \alpha}{\alpha+1-n} \int_\mu^\infty \left(\frac{\alpha+1-n}{\mu}\right)\left(\frac{x}{\mu}\right)^{-(\alpha+1-n)} dx$$

$\alpha+1-n>0$ olmalı yani $\boxed{\alpha+1>n}$ olmalı

$\alpha+1-n>0$ için

$$\underset{n=1}{E[X]} = E[X^1] = \frac{\mu \cancel{\alpha}}{\cancel{\alpha}}\int_{\mu}^{\infty}\left(\frac{\alpha}{\mu}\right)\left(\frac{x}{\mu}\right)^{-\alpha}dx$$

$$\frac{x}{\mu}=u \qquad \frac{dx}{\mu}=du$$

$$= \mu\int_{1}^{\infty}\left(\frac{\alpha}{\cancel{\mu}}\right)u^{-\alpha}\cancel{\mu}\,du = \frac{-\alpha}{\alpha-1}u^{-(\alpha-1)}\Big|_{1}^{\infty}$$

$$\checkmark\ \boxed{E[X] = \frac{\alpha}{\alpha-1}\mu}$$

$\alpha>1$ için

$$\underbrace{E[X^n]}_{\alpha>n \text{ için}} = \mu^n\int_{\mu}^{\infty}\left(\frac{\alpha}{\mu}\right)\frac{x^n}{\mu^n}\left(\frac{x}{\mu}\right)^{-(\alpha+1)}dx$$

$$= \mu^n\int_{\mu}^{\infty}\left(\frac{\alpha}{\mu}\right)\left(\frac{x}{\mu}\right)^{-(\alpha+1-n)}dx$$

$$= \frac{\mu^n\alpha}{\alpha-n}\underbrace{\int_{\mu}^{\infty}\left(\frac{\alpha-n}{\mu}\right)\left(\frac{x}{\mu}\right)^{-(\alpha+1-n)}dx}_{1}$$

$\alpha>n$ olmalı

$$\boxed{E[X^n] = \frac{\mu^n\alpha}{\alpha-n}}$$

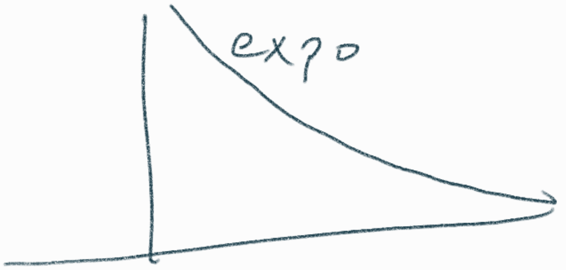
$$E[X^2] = \frac{\mu^2\alpha}{\alpha-2}$$

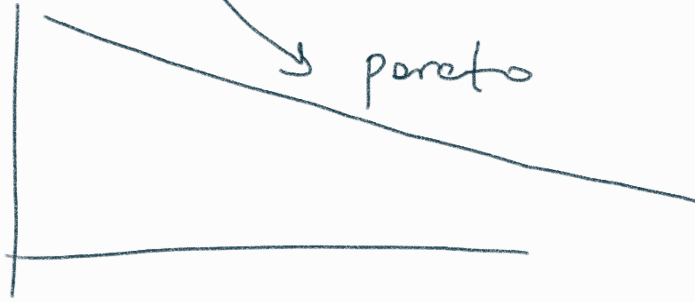
$$Var(X) = \frac{\mu^2\alpha}{\alpha-2} - \frac{\alpha^2\mu^2}{(\alpha-1)^2} = \mu^2\alpha\left(\frac{1}{\alpha-2}-\frac{\alpha}{(\alpha-1)^2}\right)$$

$$= \mu^2\alpha\left(\frac{\cancel{\alpha^2}-\cancel{2\alpha}+1-\cancel{\alpha^2}+\cancel{2\alpha}}{(\alpha-2)(\alpha-1)^2}\right)$$

$$\mathrm{Var}(X) = \frac{\mu^2 \alpha}{(\alpha-2)(\alpha-1)^2}, \quad (\alpha > 2 \text{ olmalı})$$

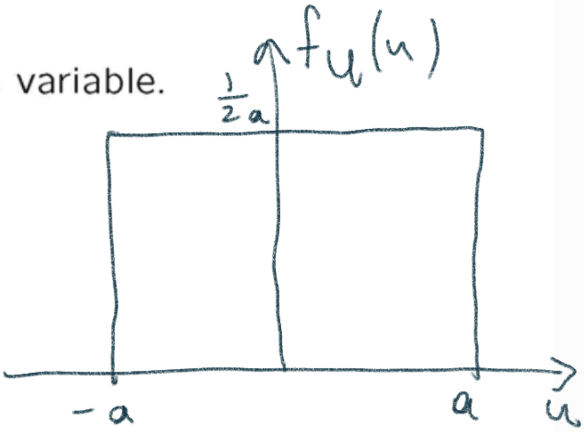
Pareto : Heavy tailed





Problem 4.5.8

U is a zero mean continuous uniform random variable.
What is P$[U^2 \leq \mathrm{Var}[U]]$?



$$\mathrm{Var}(u) = \frac{(b-a)^2}{12}$$

burada $\boxed{\mathrm{Var}(U) = \frac{a^2}{3}}$

$$P\left[U^2 \leq \mathrm{Var}(u)\right] = P\left(-\mathrm{std}(u) \leq U \leq +\mathrm{std}(u)\right)$$

$$= 2\,P\left(0 \leq U \leq \mathrm{std}(u)\right)$$

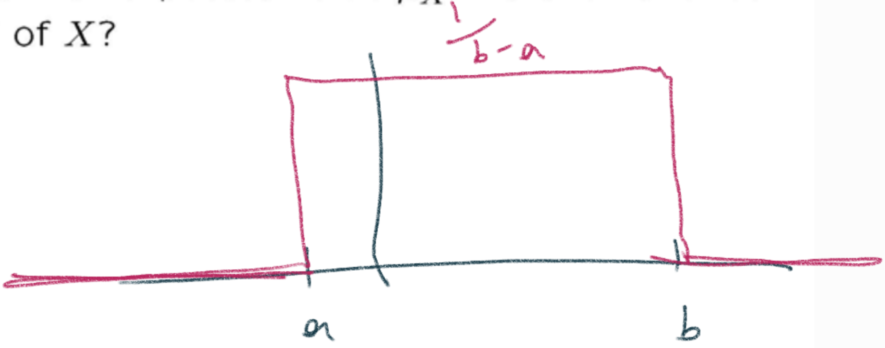
$\boxed{\mathrm{std}(u) = \frac{a}{\sqrt{3}}}$

$$= 2\,P\left(0 \leq U \leq \frac{a}{\sqrt{3}}\right)$$

$$= 2 \times \frac{a}{\sqrt{3}} \times \frac{1}{\cancel{2a}} = \frac{1}{\sqrt{3}} \;//$$

Problem 4.5.12

X is a uniform random variable with expected value $\mu_X = 7$ and variance $\text{Var}[X] = 3$. What is the PDF of X?



$$E[X] = \frac{b+a}{2}$$

$$\text{Var}[X] = \frac{(b-a)^2}{12}$$

1 $\frac{b+a}{2} = 7$

$$\boxed{b+a = 14}$$

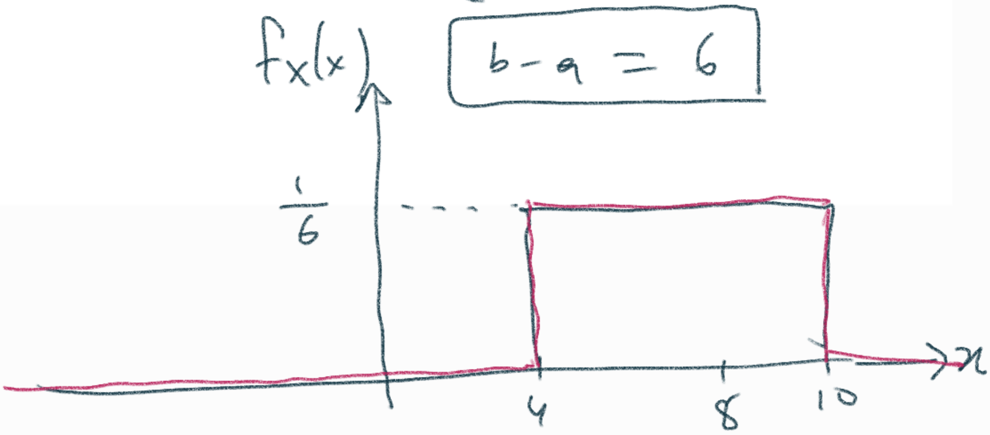
$$\boxed{b = 10}$$

$$\boxed{a = 4}$$

2 $\frac{(b-a)^2}{12} = 3$

$$(b-a)^2 = 36$$

$$\boxed{b-a = 6}$$



Problem 4.5.15

Long-distance calling plan A offers flat-rate service at 10 cents per minute. Calling plan B charges 99 cents for every call under 20 minutes; for calls over 20 minutes, the charge is 99 cents for the first 20 minutes plus 10 cents for every additional minute. (Note that these plans measure your call duration exactly, without rounding to the next minute or even second.) If your long-distance calls have exponential distribution with expected value τ minutes, which plan offers a lower expected cost per call?

$A \rightarrow 10$ ¢ / minute

$$B \rightarrow \begin{cases} 99 ¢ & T < 20 \text{ min} \\ (T-20)10+99 & T > 20 \end{cases}$$

T: duration expo$\left(\frac{1}{\tau}\right)$

$f_T(t) = \frac{1}{\tau} e^{-t/\tau}, \quad t \geq 0$

$R_A = 10 \times T$

$$R_B = \begin{cases} 99 & T < 20 \\ (T-20)10 \times 99 & T > 20 \end{cases}$$

$E[R_A] = 10\, E[T] = 10\tau$

$$E[R_B] = 99 + \underbrace{E[10(T-20)]}_{10(\tau-20)} \times \underbrace{P(T>20)}_{e^{-20/\tau}}$$

$E[R_B] = 99 + 10(\tau-20)\, e^{-20/\tau}$

$E[R_B] = \underbrace{99 - 200e^{-20/\tau}} + 10\tau$

R_A'nın daha iyi olması için $99 > 200e^{-20/\tau}$ olmalı

$\frac{99}{200} > e^{-20/\tau}$

$\ln\left(\frac{99}{200}\right) > -20/\tau$

$\ln\left(\frac{200}{89}\right) < \frac{20}{\tau} \Rightarrow \tau < \frac{20}{\ln\left(\frac{200}{89}\right)}$ olmalı

$\tau < 28.44$ olmalı

$$\boxed{\tau \underset{R_B}{\overset{R_A}{\lessgtr}} 28.44}$$

Problem 4.5.16

In this problem we verify that an Erlang (n, λ) PDF integrates to 1. Let the integral of the nth order Erlang PDF be denoted by

$$I_n = \int_0^\infty \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}\, dx.$$

First, show directly that the Erlang PDF with $n = 1$ integrates to 1 by verifying that $I_1 = 1$. Second, use integration by parts (Appendix B, Math Fact B.10) to show that $I_n = I_{n-1}$.

Problem 4.3.1

The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $P[0 \le X \le 1]$,

(c) $P[-1/2 \le X \le 1/2]$,

(d) the CDF $F_X(x)$.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_0^2 cx\,dx = \frac{cx^2}{2}\Big|_0^2$$

$$= \frac{4c}{2} - 0 = 1$$

$$\boxed{c = \frac{1}{2}}$$

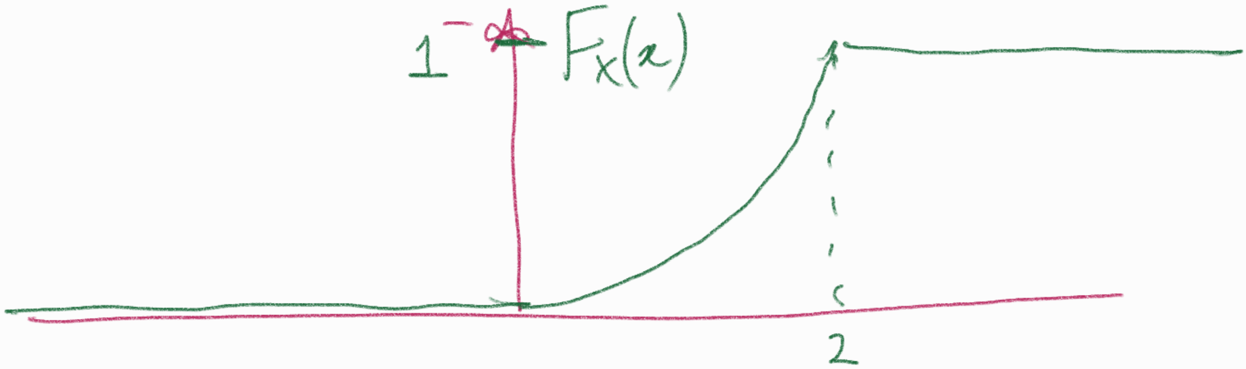
$$f_X(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0 & \text{o.w} \end{cases}$$

b) $P[0 \le X \le 1] = \int_0^1 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^1 = \frac{1}{4}$ //

c) $P[-\frac{1}{2} \le X \le \frac{1}{2}] = P[0 \le X \le \frac{1}{2}]$

$$= \int_0^{1/2} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{1/2} = \frac{1}{16}$$ //

d) $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$

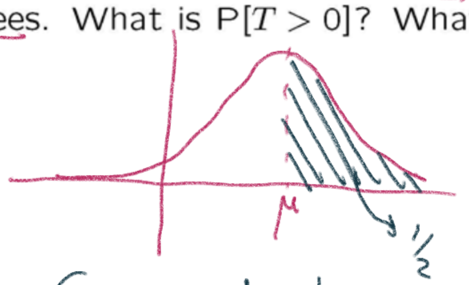


$\phi(z) = 1 - Q(z)$

Problem 4.6.9

The peak temperature T, in degrees Fahrenheit, on a July day in Antarctica is a Gaussian random variable with a variance of 225. With probability 1/2, the temperature T exceeds -75 degrees. What is P[$T > 0$]? What is P[$T < -100$]?

$\sigma = \sqrt{225} = 15 \quad T \sim \mathcal{N}(\mu, 15)$



$\mu = -75$

$T \sim \mathcal{N}(-75, 15)$

Gauss dağılımı ortalaması etrafında simetriktir.

$P(T > 0) = ?$

$= P(T + 75 > 75) = P\Big(\underbrace{\frac{T+75}{15}}_{Z \sim \mathcal{N}(0,1) \text{ standart Gauss r.d.}} > 5\Big) = Q(5) = 3 \times 10^{-7}$

1 $\left[\begin{array}{lll} X \sim & E[X] = \mu & \text{std}(X) = \sigma \\ X - \mu \rightarrow & E(X - \mu) = 0 & \text{std}(X - \mu) = \sigma \\ \frac{X-\mu}{\sigma} \rightarrow & E\left[\frac{X-\mu}{\sigma}\right] = 0 & \text{std}\left(\frac{X-\mu}{\sigma}\right) = 1 \end{array}\right]$ Bütün dağılımlar için geçerli

2 $X \sim \mathcal{N} \iff aX + b \sim \mathcal{N}$

b) $P(T < -100) = P(T - (-75)) < -25)$

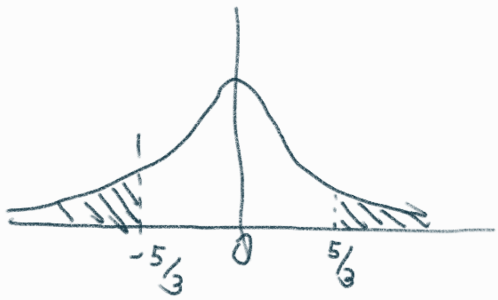
$= P\Big(\underbrace{\frac{T+75}{15}}_{Z \sim \mathcal{N}(0,1)} < \frac{-25}{15}\Big) = \phi\left(-\frac{5}{3}\right)$

$= Q\left(\frac{5}{3}\right)$

$= 1 - \phi\left(\frac{5}{3}\right)$

$= 1 - 0.9525$

$= 0.0475$



Problem 4.6.11

Suppose that out of 100 million men in the United States, 23,000 are at least 7 feet tall. Suppose that the heights of U.S. men are independent Gaussian random variables with a expected value of $5'10''$. Let N equal the number of men who are at least $7'6''$ tall.

(a) Calculate σ_X, the standard deviation of the height of U.S. men.

(b) In terms of the $\Phi(\cdot)$ function, what is the probability that a randomly chosen man is at least 8 feet tall?

(c) What is the probability that no man alive in the United States today is at least $7'6''$ tall?

(d) What is $E[N]$?

7 feet = 210cm

$\mu = 5''10'' = 150 + 25.4 = 174$ cm

$7'6'' = 210 + 6\times 2.54 = 225$ cm

H : boy r.d.

(1) $P(H > 210) = \dfrac{23\cancel{000}}{100\,000\,\cancel{000}} = 2.3\times10^{-4}$

(2) $\mu = 174$cm

$Z \sim \mathcal{N}(0,1)$

$$P(H > 210) = \underbrace{P\left(\frac{H-174}{\sigma_H} > \frac{210-174}{\sigma_H}\right)}\, = 2.3\times10^{-4}$$

$$= Q\left(\frac{36}{\sigma_H}\right) = 2.3\times10^{-4}$$

$$= \frac{36}{\sigma_H} \simeq 3.5 \Rightarrow \sigma_H = \frac{36}{3.5} = 10.3 \text{ cm}$$

b) $P(H > 240)$

$$= P\left(\frac{H-\mu}{\sigma} > \frac{240-\mu}{\sigma}\right) = P\left(Z > \frac{240-174}{10.3}\right) = Q(6.41)$$

$$= 1 - \Phi(6.41)$$

c) $P\{$no man alive taller than 225cm$\}$

$= P\{$all man alive shorter than 225cm$\}$

$= p^{100\,000\,000}$

$p = P\{$a man is shorter than 225cm$\} = P(H < 225)$

$$p = P\left(\frac{H-174}{10.3} < \frac{225-174}{10.3}\right) = \phi(4.95) = 1 - Q(4.95)$$
$$= 1 - 3.71\times10^{-7}$$

istenen olasılık değeri $= \left(1-3.71\times10^{-7}\right)^{100000000}$

d) $N \sim \text{Binomial}\left(\underbrace{100\,000\,000}_{n}, \underbrace{3.71\times10^{-7}}_{p}\right)$

$E[N] = np$

$= 10^{8} \times 3.71\times10^{-7} = 37.1$ adam

k

Problem 4.6.14

hisse senedi

At time $t = 0$, the price of a stock is a constant k dollars. At time $t > 0$ the price of a stock is a Gaussian random variable X with $E[X] = k$ and $\text{Var}[X] = t$. At time t, a *Call Option at Strike* k has value $V = (X-k)^+$, where the operator $(\cdot)^+$ is defined as $(z)^+ = \max(z, 0)$.

$$V = \begin{cases} X-k & , \text{if } X > k \\ 0 & , \text{if } X \leq k \end{cases}$$

(a) Find the expected value $E[V]$.

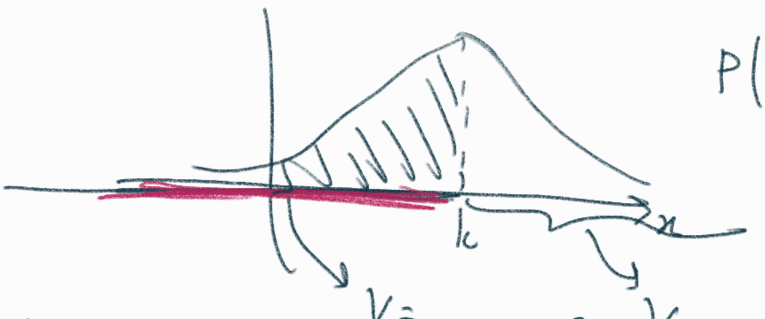
(b) Suppose you can buy the call option for d dollars at time $t = 0$. At time t, you can sell the call for V dollars and earn a profit (or loss perhaps) of $R = V - d$ dollars. Let d_0 denote the value of d such that $P[R > 0] = 1/2$. Your strategy is that you buy the option if $d \leq d_0$ so that your probability of a profit is $P[R > 0] \geq 1/2$. Find d_0.

(c) Let d_1 denote the value of d such that $E[R] = 0.01 \times d$. Now your strategy is to buy the option if $d \leq d_1$ so that your expected return is at least one percent of the option cost. Find d_1.

(d) Are the strategies "Buy the option if $d \leq d_0$" and "Buy the option if $d \leq d_1$" reasonable strategies?

a) $E[V] = \int_0^{\infty} f_V(v)\, v\, dv$

$V \in [0, \infty)$

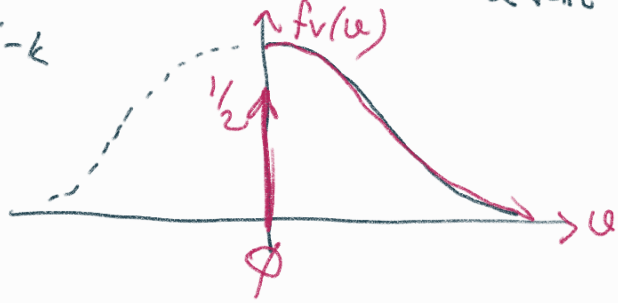


$P(V=0) = 1/2$

$V \sim$ karma rastgele değişken

$$f_V(v) = \frac{1}{2}\delta(v) + \frac{1}{2}\frac{1}{\sqrt{2\pi t}} e^{-\frac{v^2}{2t}}, \quad v \geq 0$$

$$E[V] = \underbrace{\int_0^{0^+} \frac{1}{2}\delta(v)\, v\, dv}_{0} + \int_0^{\infty} v \frac{1}{\sqrt{2\pi t}} e^{-v^2/2t}\, dv$$



$$E[V] = \int_0^\infty v \frac{1}{\sqrt{2\pi t}} e^{-v^2/2t}\, dv$$

$$v^2/2t = u \Rightarrow \frac{v}{t} dv = du$$

$$v\, dv = t\, du$$

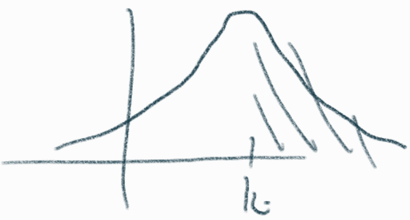
$$E[V] = \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-u}\, t\, du = \sqrt{\frac{t}{2\pi}} \underbrace{\int_0^\infty \underbrace{e^{-u}}_{\text{expo(1) pdf}}\, du}_{1}$$

o nedenle integral

$$\boxed{E[V] = \sqrt{\frac{t}{2\pi}}}$$

b) $P(R > 0) = P(V - d_0 > 0) \geq 1/2$

$$= P(\underbrace{V}_{(X-k)^+} > d_0) \geq \frac{1}{2}$$



$$= P(X > k + d_0) \geq \frac{1}{2}$$

$\boxed{d_0 = 0}$ $\quad d_0 < 0$

c) $R = V - d_1$

$R = (X-k)^+ - d_1$

$$E[R] = \underbrace{E[(X-k)^+]}_{\sqrt{\frac{t}{2\pi}}} - d_1 = \underline{0.01 d_1}$$

$$= \sqrt{\frac{t}{2\pi}} > 1.01 d_1 \Rightarrow d_1 = \frac{1}{1.01}\sqrt{\frac{t}{2\pi}} \text{ olmalı.}$$

d) d_0 opsiyonun mantıklı değil çünkü $d_0 \leq 0$ olmalı

d_1 daha mantıklı ama modelin doğru olması lazım.

Problem 4.6.15

In mobile radio communications, the radio channel can vary randomly. In particular, in communicating with a fixed transmitter power over a "Rayleigh fading" channel, the receiver signal-to-noise ratio Y is an exponential random variable with expected value γ. Moreover, when $Y = y$, the probability of an error in decoding a transmitted bit is $P_e(y) = Q(\sqrt{2y})$ where $Q(\cdot)$ is the standard normal complementary CDF. The average probability of bit error, also known as the bit error rate or BER, is

$$\overline{P}_e = \mathrm{E}[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy.$$

Find a simple formula for the BER $\overline{P}_e$ as a function of the average SNR γ.

$$Y \sim \text{expo}(\gamma) \Rightarrow f_Y(y) = \frac{1}{\gamma} e^{-y/\gamma}, \quad y \geq 0$$

$$E[Y] = \gamma$$

$$Y = y \rightarrow P_e(y) = Q(\sqrt{2y})$$

$$E[P_e(Y)] = \int_{y=0}^{\infty} \underbrace{Q(\sqrt{2y})} \frac{1}{\gamma} e^{-y/\gamma}\, dy \Rightarrow \text{ortalama hata oranı}$$

$$\left(Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)$$

$$Q(\sqrt{2y}) = \int_{\sqrt{2y}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$\overline{P}_e = \int_{y=0}^{\infty} \left(\int_{x=\sqrt{2y}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right) \frac{y}{\gamma} e^{-y/\gamma} dy$$

$$Q(\sqrt{2y}) = \int_{\sqrt{2y}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = u$$

$$\boxed{\begin{gathered} \frac{1}{\gamma} e^{-y/\gamma} dy = dv \\ -e^{-y/\gamma} = v \end{gathered}}$$

$$\frac{-1}{\sqrt{2}\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y} dy = du$$

$$\frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

$$\int u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= \underbrace{-Q(\sqrt{2y})\,e^{-y/\gamma}\Big|_{y=0}^{\infty}}_{1/2} - \int_0^\infty -e^{-y/\gamma}\frac{1}{\sqrt{2y}}\frac{1}{\sqrt{2\pi}}e^{-y}\,dy$$

$$\boxed{\overline{P_e} = \frac{1}{2} - \frac{1}{2}\frac{\gamma}{1+\gamma}} = \frac{1}{2}\frac{1}{1+\gamma}$$

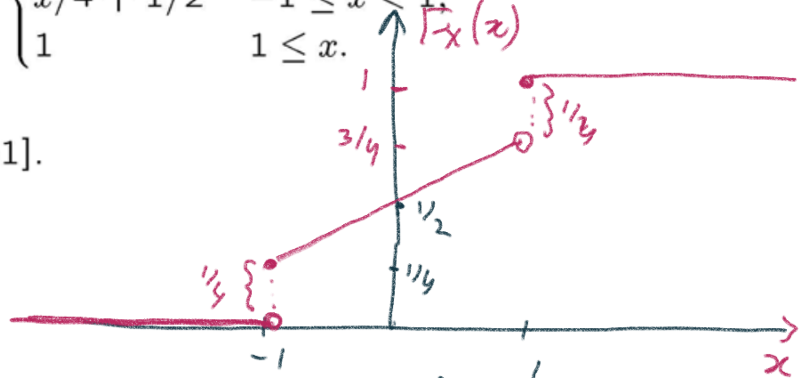
Problem 4.7.2

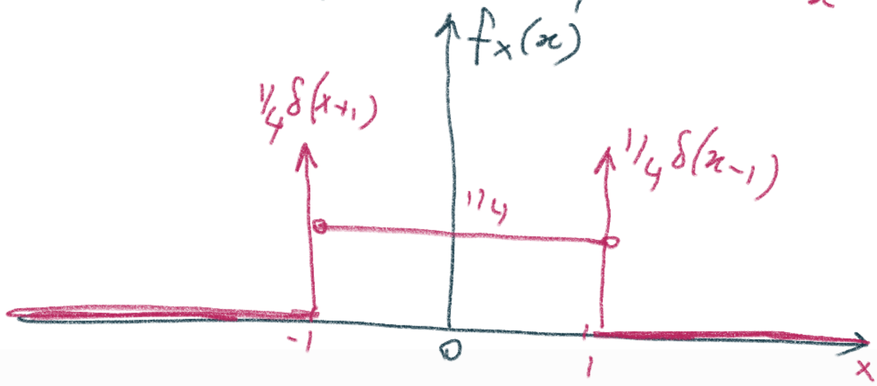
Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 0 & x < -1, \\ x/4 + 1/2 & -1 \le x < 1, \\ 1 & 1 \le x. \end{cases}$$

Sketch the CDF and find

(a) $P[X < -1]$ and $P[X \le -1]$.
(b) $P[X < 0]$ and $P[X \le 0]$.
(c) $P[X > 1]$ and $P[X \ge 1]$.





a) $P(X<-1) = 0$

$P(X \le -1) = 1/4$

b) $P(X<0) =$

$$= \int_{-\infty}^{0} f_X(x)\,dx$$

$$= \int_{-\infty}^{-1^+} \frac{1}{4}\delta(x+1) + \int_{-1^+}^{0} \frac{1}{4}\,dx$$

$$= \frac{1}{4} + \frac{1}{4} = 1/2$$

KARMA R.D.

$P(X \le 0) = 1/2$

c) $P(X>1) = 0$

$P(X \ge 1) = 1/4$

4.5.20

$$E[X]=\int_0^\infty x f_X(x)\,dx \geq \int_r^\infty x f_X(x)\,dx \geq \int_r^\infty r f(x)\,dx$$

$$= r\,P(X \geq r)$$

$$\int_0^\infty x f_X(x)\,dx$$

$$\boxed{\begin{aligned} f_X(x)\,dx &= dv \\ F_X(x) &= v \\ x &= u \\ dx &= du \end{aligned}}$$

$$\int u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= \underbrace{x F_X(x)\Big|_0^\infty}_{=\int_0^\infty 1\,dx} - \int_0^\infty F_X(x)\,dx$$

$$\boxed{E[X] = \int_0^\infty 1 - F_X(x)\;dx}$$

4.6.13

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(\omega-\mu)^2/2\sigma^2}\, d\omega$$

$$x = \frac{\omega-\mu}{\sigma}$$

$$dx = \frac{+d\omega}{\sigma}$$

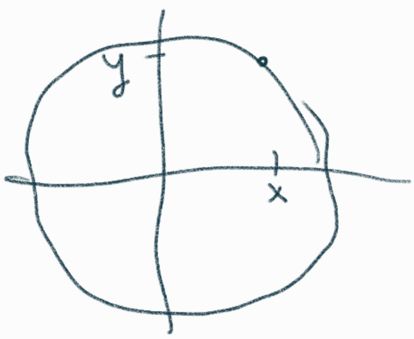
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}}\, e^{-x^2/2}\, \cancel{\sigma}\, dx$$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}\, dx\,dy$$

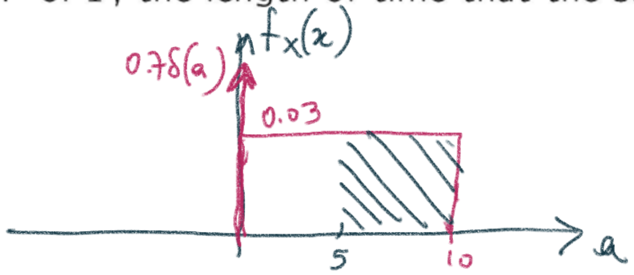
$$x = r\cos\theta$$
$$y = r\sin\theta$$



Problem 4.7.9

For 70% of lectures, Professor Y arrives on time. When Professor Y is late, the arrival time delay (in minutes) is a continuous uniform $(0, 10)$ random variable. Yet, as soon as Professor Y is 5 minutes late, all the students get up and leave. If a lecture starts when Professor Y arrives and always ends 80 minutes after the scheduled starting time, what is the PDF of T, the length of time that the students observe a lecture.

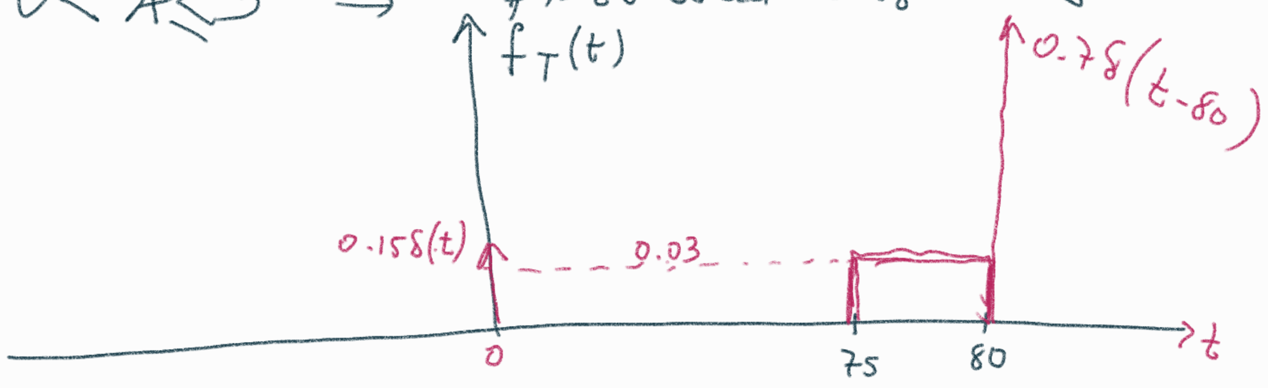
A.



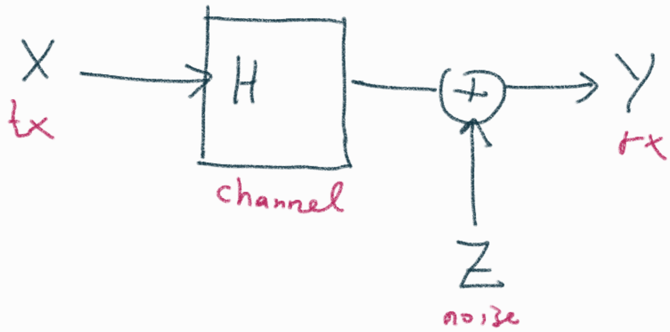
$$P(A>5) = P(T=0) = 0.15 = \int_5^\infty f_X(x)\,dx$$

$$P(A=0) = P(T=80) = 0.7$$

$0 < A \leq 5 \rightarrow$ 75–80 arası düzgün dağılımlı



CHAPTER FIVE : MULTIPLE RANDOM VARIABLES:



X, Y, Z

bağımsız

$Y = XH + Z$

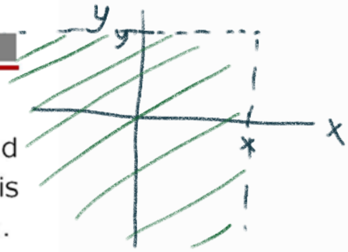
$Y = X + Z$

(X, Y), (X, Z), (Y, Z)

$f_{XY}(x,y)$ $f_{XZ}(x,z)$

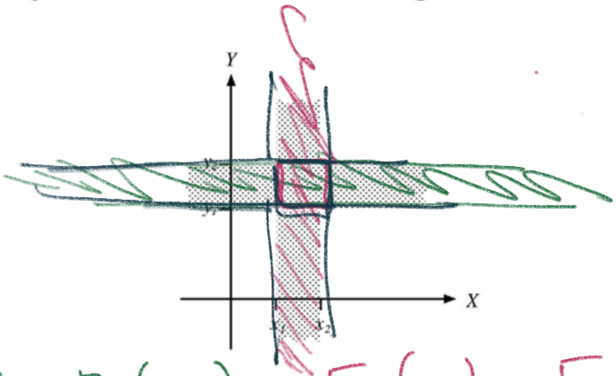
$$\sum_{y \in S_Y} P_{XY}(x,y) = P_X(x)$$

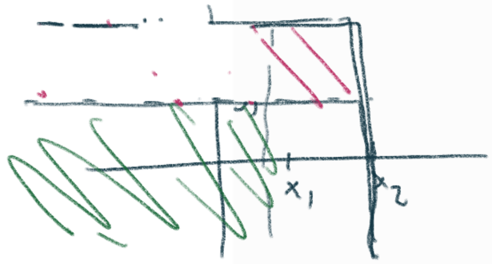
Problem 5.1.3



$F_{X,Y}(x,y) = P[X \le x, Y \le y]$

For continuous random variables X, Y with joint CDF $F_{X,Y}(x,y)$ and marginal CDFs $F_X(x)$ and $F_Y(y)$, find P$[x_1 \le X < x_2 \cup y_1 \le Y < y_2]$. This is the probability of the shaded "cross" region in the following diagram.





$$F_Y(y_2) - F_Y(y_1) + F_X(x_2) - F_X(x_1) - \underbrace{P(x_1 < X \le x_2 \cap y_1 < Y \le y_2)}_{?}$$

$F_{XY}(x_2,y_2) - F_{XY}(x_2,y_1)$
$- F_{XY}(x_1,y_2)$
$+ F_{XY}(x_1,y_1)$

$$F_Y(y_2) - F_Y(y_1) + F_X(x_2) - F_X(x_1)$$
$$- F_{XY}(x_2,y_2) + F_{XY}(x_2,y_1)$$
$$+ F_{XY}(x_1,y_2) - F_{XY}(x_1,y_1)$$

Problem 5.1.4

Random variables X and Y have CDF $F_X(x)$ and $F_Y(y)$. Is $F(x,y) = F_X(x)F_Y(y)$ a valid CDF? Explain your answer.

$\boxed{F_{XY}(x,y) = F_X(x)F_Y(y)}$ geçerli bir CDF' mi?

a) $0 \le F_{XY}(x,y) \le 1$ ✓ çünkü $0 \le F_X \le 1$ ve $0 \le F_Y(y) \le 1$

b) $F_{X,Y}(\infty,\infty) = 1$ ✓ $F_{XY}(\infty,\infty) = \underbrace{F_X(\infty)}_{1}\underbrace{F_Y(\infty)}_{1} = 1$

c) $F_X(x) = F_{XY}(x,\infty)$ ✓ $F_{XY}(x,\infty) = F_X(x)\underbrace{F_Y(\infty)}_{1} = F_X(x)$

d) $F_Y(y) = F_{XY}(\infty,y)$ ✓

e) $F_{XY}(x,-\infty) = 0$ ✓ $F_X(x)\underbrace{F_Y(-\infty)}_{0} = 0$

f) $F_{XY}(-\infty,y)$ ✓

g) $x \le x_1$ ve $y \le y_1$ için $F_{XY}(x,y) \le F_{XY}(x_1,y_1)$

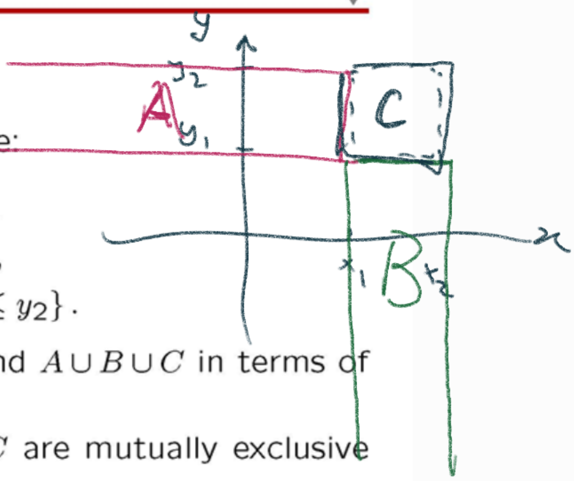
$F_X(x)F_Y(y) \le F_X(x_1)F_Y(y_1)$

Problem 5.1.5

In this problem, we prove Theorem 5.2.

✓ (a) Sketch the following events on the X,Y plane:

$$A = \{X \le x_1, y_1 < Y \le y_2\},$$
$$B = \{x_1 < X \le x_2, Y \le y_1\},$$
$$C = \{x_1 < X \le x_2, y_1 < Y \le y_2\}.$$



✓ (b) Express the probability of the events A, B, and $A \cup B \cup C$ in terms of the joint CDF $F_{X,Y}(x,y)$.

(c) Use the observation that events A, B, and C are mutually exclusive to prove Theorem 5.2.

Theorem 5.2: $P(x_1 < X \le x_2, y_1 < Y \le y_2) = F_{XY}(x_2,y_2) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2) + F_{XY}(x_1,y_1)$

$P(A) = F_{XY}(x_1,y_2) - F_{XY}(x_1,y_1)$

$P(B) = F_{XY}(x_2,y_1) - F_{XY}(x_1,y_1)$

$P(A \cup B \cup C) = F_{XY}(x_2,y_2) - F_{XY}(x_1,y_1)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ çünkü A, B ve C ayrık olaylar

$P(C) = P(A \cup B \cup C) - P(A) - P(B)$

$= F_{XY}(x_2,y_2) - \cancel{F_{XY}(x_1,y_1)} - F_{XY}(x_1,y_2) + F_{XY}(x_1,y_1) - F_{XY}(x_2,y_1) + \cancel{F_{XY}(x_1,y_1)}$

$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$ //

Problem 5.1.6

Can the following function be the joint CDF of random variables X and Y?

$$F(x, y) = \begin{cases} 1 - e^{-(x+y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

a) $0 \leq F(x, y) \leq 1$ ✓

b) $F(\infty, \infty) = 1$ ✓

c) $F(x, \infty) \neq F_X(x)$ X

d) $F(\infty, x) \neq F_Y(y)$ X

e) $F(x, -\infty) = 0$ ✓

f) $F(-\infty, y) = 0$ ✓

geçerli bir CDF olamaz.

$F_{XY}(x, \infty) = 1 - e^{-(x+\infty)} = 1$

$F_X(x)$

$f_X(x)$

Problem 5.2.1

ortaklaşa olasılık ağırlık fonksiyonu

Random variables X and Y have the joint PMF

$$P_{X,Y}(x, y) = \begin{cases} cxy & x = 1, 2, 4; \quad y = 1, 3, \\ 0 & \text{otherwise.} \end{cases}$$

✓(a) What is the value of the constant c?

(b) What is $P[Y < X]$?

(c) What is $P[Y > X]$?

(d) What is $P[Y = X]$?

(e) What is $P[Y = 3]$?

Y \ X	1	2	4
1	c	2c	4c
3	3c	6c	12c

→ $Y=3$

a) $P_{XY}(x, y) = P(X = x, Y = y)$

$c + 2c + 4c + 3c + 6c + 12c = 1$ olmalı

$c = 1/28$ olmalı

b) $P[Y < X] = 2c + 4c + 12c = \frac{18}{28}$

c) $P[Y > X] = 3c + 6c = \frac{9}{28}$

d) $P[Y = X] = c = 1/28$

e) $P[Y = 3] = 3c + 6c + 12c = \frac{21}{28}$

Problem 5.2.4

For two independent flips of a fair coin, let X equal the total number of tails and let Y equal the number of heads on the last flip. Find the joint PMF $P_{X,Y}(x,y)$.

2 bağımsız atış

$S_X = \{0,1,2\}$

$S_Y = \{0,1\}$

X : Tura sayısı

Y : son atıştaki yazı sayısı

Y \ X	0	1	2
0	0	1/4	1/4
1	1/4	1/4	0

Çıktılar $\{YY, YT, TY, TT\}$

$\{X=0, Y=0\} = \{YY\} \cap \{YT, TT\} = \phi$

$\{X=1, Y=0\} = \{YT, TY\} \cap \{YT, TT\} = \{YT\}$

$\{X=2, Y=0\} = \{TT\} \cap \{YT, TT\} = \{TT\}$

$\{X=0, Y=1\} = \{YY\} \cap \{YY, TY\} = \{YY\}$

$\{X=1, Y=1\} = \{YT, TY\} \cap \{YY, TY\} = \{TY\}$

$\{X=2, Y=1\} = \{TT\} \cap \{YY, TY\} = \phi$

$$P_{XY}(x,y) = \begin{cases} 1/4 & x=1, y=0 \\ 1/4 & x=2, y=0 \\ 1/4 & x=0, y=1 \end{cases}$$

Problem 5.2.7

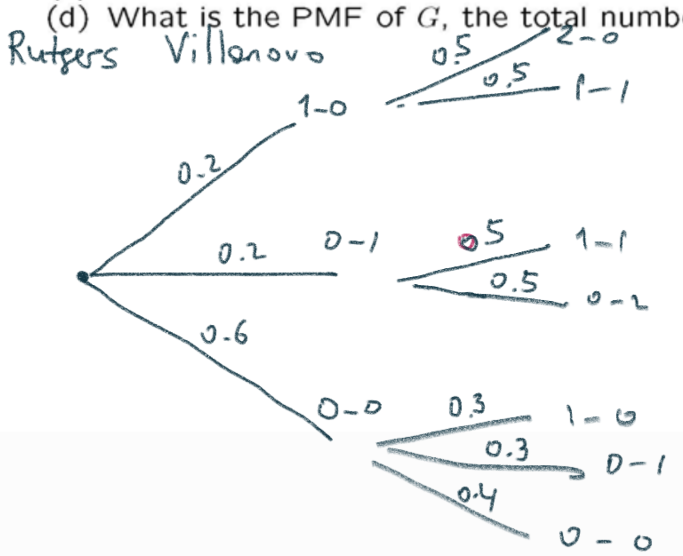
With two minutes left in a five-minute overtime, the score is 0–0 in a Rutgers soccer match versus Villanova. (Note that the overtime is NOT *sudden-death*.) In the next-to-last minute of the game, either (1) Rutgers scores a goal with probability $p = 0.2$, (2) Villanova scores with probability $p = 0.2$, or (3) neither team scores with probability $1-2p = 0.6$. If neither team scores in the next-to-last minute, then in the final minute, either (1) Rutgers scores a goal with probability $q = 0.3$, (2) Villanova scores with probability $q = 0.3$, or (3) neither team scores with probability $1-2q = 0.4$. However, if a team scores in the next-to-last minute, the trailing team goes for broke so that in the last minute, either (1) the leading team scores with probability 0.5, or (2) the trailing team scores with probability 0.5. For the final two minutes of overtime:

(a) Sketch a probability tree and construct a table for $P_{R,V}(r,v)$, the joint PMF of R, the number of Rutgers goals scored, and V, the number of Villanova goals scored.

(b) What is the probability P[T] that the overtime ends in a tie?

(c) What is the PMF of R, the number of goals scored by Rutgers?

(d) What is the PMF of G, the total number of goals scored?



V \ R	0	1	2
0	0.24	0.18	0.1
1	0.18	0.2	0
2	0.1	0	0

b) $P[T] = P[R=V] = P[0,0) + P(1,1) = 0.24 + 0.2 = 0.44$

c) $P_R(r)$?

$P_R(0)$, $P_R(1)$, $P_R(2)$

0.52 $\quad$ 0.38 $\quad$ 0.1

$$P_R(r) = \begin{cases} 0.52 & r=0 \\ 0.38 & r=1 \\ 0.1 & r=2 \\ 0 & \text{o.w} \end{cases}$$

d) $G = R+V$

$S_G = \{0,1,2\}$

$P_G(0)$, $P_G(1)$, $P_G(2)$?

0.24 $\quad$ 0.36 $\quad$ 0.4

$$P_G(r) = \begin{cases} 0.24 & g=0 \\ 0.36 & g=1 \\ 0.4 & g=2 \\ 0 & \text{o.w.} \end{cases}$$

Problem 5.2.9

Each test of an integrated circuit produces an acceptable circuit with probability p, independent of the outcome of the test of any other circuit. In testing n circuits, let K denote the number of circuits rejected and let X denote the number of acceptable circuits that appear before the first reject is found. Find the joint PMF $P_{K,X}(k,x)$.

$K \sim \text{Binom}(n, 1-p)$

$f_X(x) = p^x(1-p)$, $x = 0,1,2,\ldots$

$P_{K,X}(0,0) = 0$, $P_{KX}(1,0) = (1-p)\,p^{n-1}$, $P_{KX}(2,0) = (1-p)\binom{n-1}{1}(1-p)\,p^{n-2}$

- $\boxed{P_{KX}(k,0) = (1-p)\binom{n-1}{k-1}(1-p)^{k-1}\,p^{n-k}, \quad k = 1,2,\ldots,n}$
- $P_{KX}(0,x) = 0$, $x = 1,2,3,\ldots,n-1$
- $P_{KX}(0,n) = p^n$
- $P_{KX}(1,1) = p(1-p)\,p^{n-2} = (1-p)\,p^{n-1}$ ✓

$P_{KX}(1,2) = p^2(1-p)\,p^{n-3} = (1-p)\,p^{n-1}$

- $\boxed{P_{KX}(1,x) = (1-p)\,p^{n-1}, \quad x = 0,1,\ldots,n-1}$

$P_{KX}(2,1) = p(1-p)\binom{n-2}{1}(1-p)\,p^{n-3} = p^{n-2}(1-p)^2\binom{n-2}{1}$

$P_{KX}(3,1) = p(1-p)\binom{n-2}{2}(1-p)^2\,p^{n-4} = p^{n-3}(1-p)^3\binom{n-2}{2}$

$$\boxed{P_{KX}(k,1) = p^{n-k}(1-p)^{k}\binom{n-2}{k-1}, \quad k = 1, 2, \dots, n-1}$$

$$P_{KX}(2,2) = p^{2}(1-p)\binom{n-2}{1}(1-p)\,p^{n-3}$$

$$P_{KX}(k,x) = p^{x}(1-p)\binom{n-x-1}{k-1}(1-p)^{k-1}\,p^{n-x-k}, \quad k = 1, 2, \dots \quad x = 1, 2, \dots \quad k + x = n$$

$$= (1-p)^{k}\,p^{n-k}\binom{n-x-1}{k-1}$$

$k=1, x=1 \longrightarrow (1-p)\,p^{n-1}\binom{n-2}{0}$

$k=n \longrightarrow (1-p)^{n}\binom{n-x-1}{n-1}$

$k=0, x=n$

$p^{n}(1-p)$

$1-p\binom{n-1}{n-1}(1-p)^{n-1}$

$$\boxed{P_{KX}(k,x) = p^{n} \quad k=0, \; x=n}$$

$P_{KX}(k,x) \quad (1-p)^{n} \qquad x=0, \; k=n$

$$\boxed{P_{KX}(k,x) = (1-p)^{k}\,p^{n-k}\binom{n-x-1}{k-1} \qquad k \geq 0, \; x \geq 0, \; k+x \leq n}$$