CH 16 LAPLACE DÖNÜŞÜMÜ UYGULAMALAR

16.1 Giriş

- Sistem : Girdi ve çıktıyı ilişkilendiren fiziksel bir sürecin matematiksel modeli.
- Laplace dönüşümü sistem analizindeki en kullanışlı araçlardan biridir.
- Devreler de bir sistemdir.

Laplace D. ile devre analizi

1. Devreyi zaman alanından S-alanına çevirmek
2. S-alanında devreyi çözmek (D.G.Y. , Ç.A.Y.)
3. Ters Laplace dönüşümü yapmak.

16.2 Devre Eleman Modelleri:

Direnç:

$$v(t) = R\, i(t)$$

$$\Big\downarrow \mathcal{L}$$

$$V(s) = R\, I(s)$$

S-alanındaki eşdeğeri yine R'dir.

Endüktör:

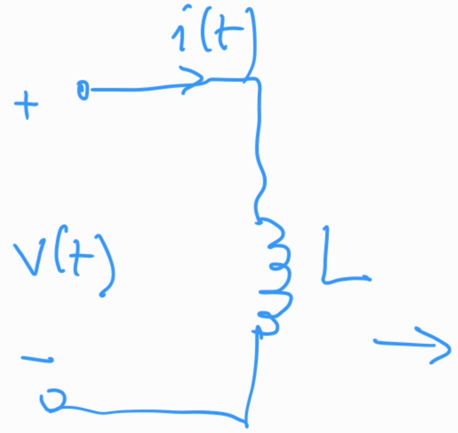
$$v(t) = L \frac{d\, i(t)}{dt}$$

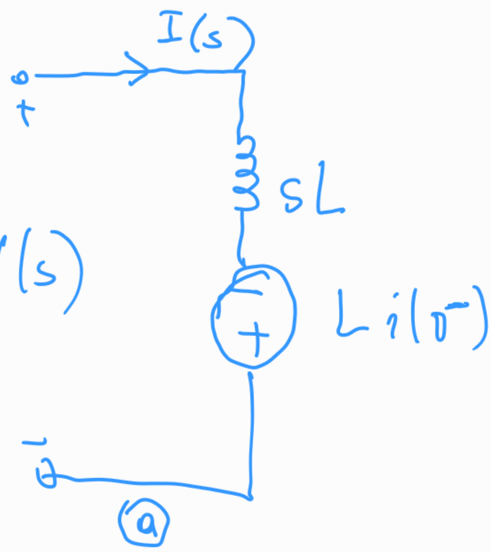
$$\Big\downarrow \mathcal{L}$$

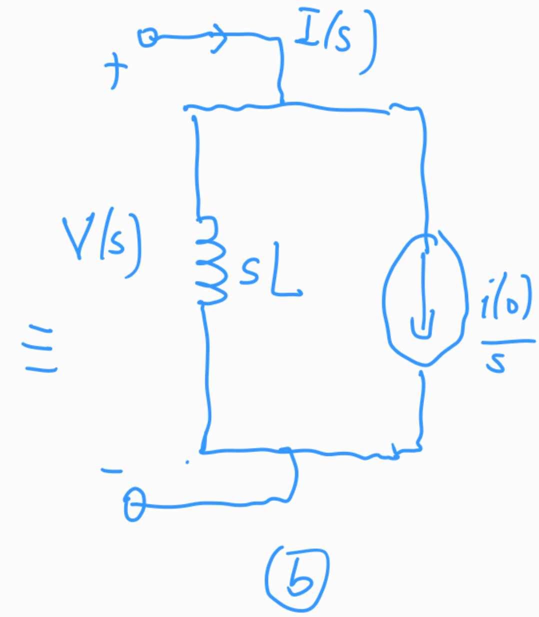
$$V(s) = L\left[s I(s) - i(0^-) \right]$$

$$\longrightarrow V(s) = sL\, I(s) - L\, i(0^-)$$

sL büyüklüğünde empedans ve ona seri halde $-L i(0^-)$ büyüklüğünde bir gerilim kaynağı.







Diğer bir ifade:

$$I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s}$$

sL empedansı ve ona paralel halde $\frac{i(0^-)}{s}$ akım kaynağı olarak modellenebilir

Hangı modelin kullanılacağı devrenin şekline göre değişir.

Kondansatör:

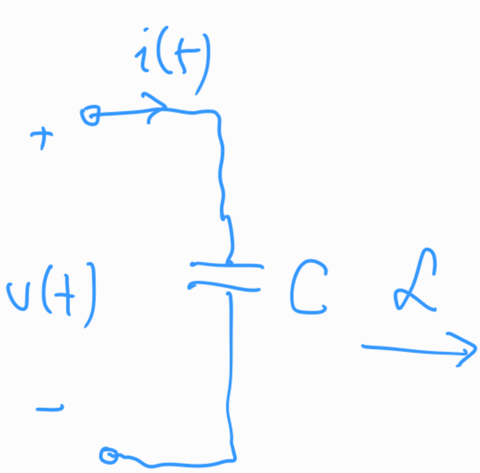
$$i(t) = C\,\frac{dv(t)}{dt}$$

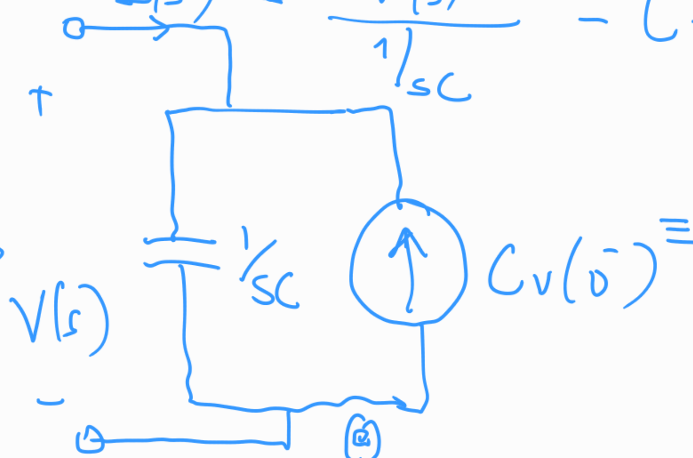
$$\downarrow \mathcal{L}$$

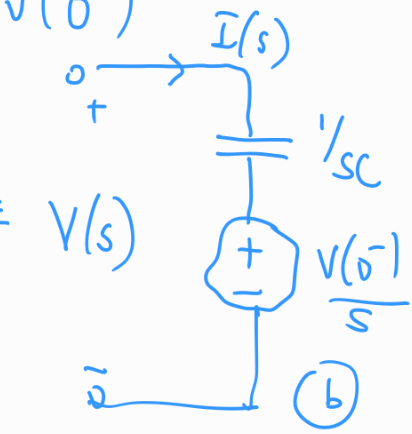
$$I(s) = C\left[sV(s) - v(0^-)\right]$$

$$\Rightarrow I(s) = sC\,V(s) - C\,v(0^-)$$

$$I(s) = \frac{V(s)}{1/sC} - Cv(0^-)$$







$$V(s) = \frac{I(s)}{sC} + \frac{\cancel{C}\,v(0^-)}{s\cancel{C}}$$

- Başlangıç koşulları modelin bir parçası
- Hem geçici, hem de yatışkın durum tepkisi gözlenebilir.

Example 16.1

Find $v_o(t)$ in the circuit of Fig. 16.4, assuming zero initial conditions.

Solution:

We first transform the circuit from the time domain to the s-domain.

$$u(t) \Rightarrow \frac{1}{s}$$

$$1\text{ H} \Rightarrow sL = s$$

$$\frac{1}{3}\text{F} \Rightarrow \frac{1}{sC} = \frac{3}{s}$$

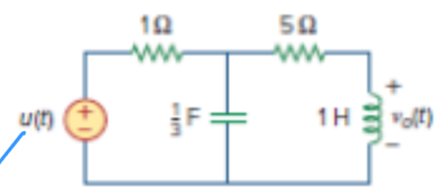
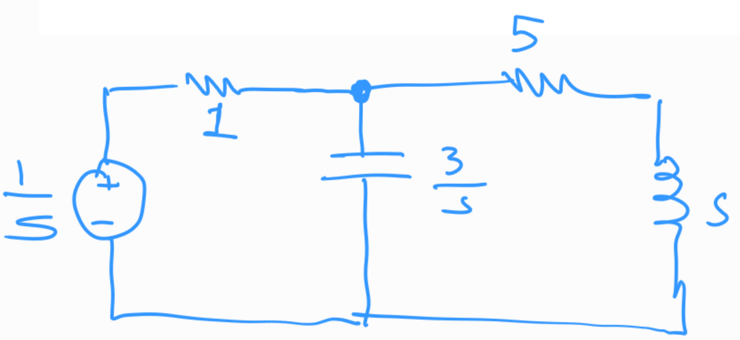


Figure 16.4
For Example 16.1.



\+ $V_o(s)$? −

$$V_o(s) = \frac{\frac{3}{s} \,//\, (5+s)}{1 + \left(\frac{3}{s} \,//\, (5+s)\right)} \times \frac{s}{5+s} \times \frac{1}{s}$$

$$= \frac{\dfrac{\frac{3}{s}(5+s)}{\frac{3}{s} + 5 + s}}{1 + \dfrac{\frac{3}{s}(5+s)}{\frac{3}{s} + 5 + s}} \times \frac{s}{5+s} \times \frac{1}{s}$$

$$= \frac{\frac{3}{\cancel{s}}(\cancel{5+s})}{\frac{3}{s} + 5 + s + \frac{3}{s}(5+s)} \times \frac{\cancel{s}}{\cancel{5+s}} \times \frac{1}{s}$$

$$= \frac{1}{s}\,\frac{3}{\frac{3}{s} + 5 + s + \frac{15}{s} + 3} = \frac{3}{s + 8 + \frac{18}{s}}\,\frac{1}{s}$$

$$\boxed{V_o(s) = \frac{3}{s^2 + 8s + 18}}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$$

$$V_o(s) = \frac{3\sqrt{2}/\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2} = \frac{3}{\sqrt{2}}\,\frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

$$v_o(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin(\sqrt{2}\,t)\,u(t)$$

Practice Problem 16.1 Determine $v_o(t)$ in the circuit of Fig. 16.6, assuming zero initial conditions.

Answer: $10(1 - e^{-2t} - 2te^{-2t})u(t)$ V.

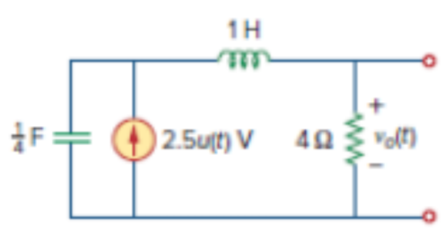
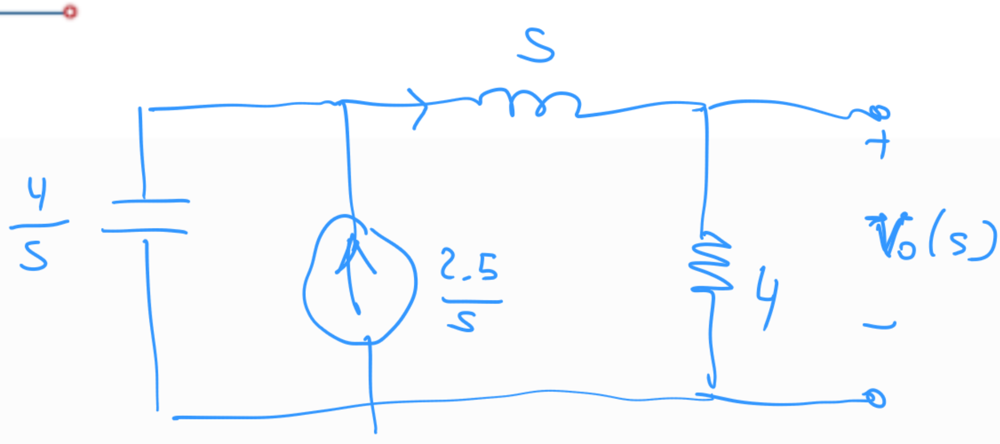


Figure 16.6
For Practice Prob. 16.1.

s-alanı;



$$V_o(s) = \frac{2.5}{s} \times \frac{\frac{4}{s}}{\frac{4}{s} + s + 4} \times 4$$

$$= \frac{10}{s} \times \frac{4}{s^2 + 4s + 4} = \frac{40}{s(s+2)^2}$$

$$V_o(s) = \frac{\overset{10}{K_1}}{s} + \frac{\overset{-20}{K_2}}{(s+2)^2} + \frac{\overset{10}{K_3}}{s+2}$$

$$K_1 = \left.\frac{40}{(s+2)^2}\right|_{s=0} = 10 \qquad K_2 = \left.\frac{40}{s}\right|_{s=-2}$$

$$K_3 = \left.\frac{d}{ds}\left[\frac{40}{s}\right]\right|_{s=-2} = \frac{-40}{s^2} = -10$$

$$V_o(s) = \frac{10}{s} - \frac{20}{(s+2)^2} - \frac{10}{s+2}$$

$$v_o(t) = \left(10 - 20te^{-2t} - 10e^{-2t}\right) u(t)$$

$V_o(0) = 0$ ✓

$V_o(\infty) = 10$ V

$\delta(t) \quad 1$

Example 16.2 Find $v_o(t)$ in the circuit of Fig. 16.7. Assume $v_o(0) = 5$ V.

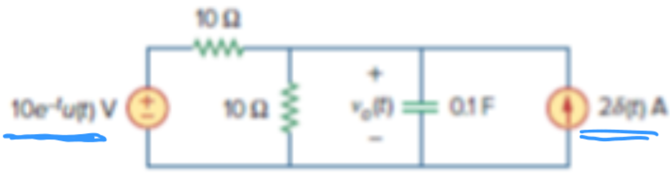


Figure 16.7
For Example 16.2.

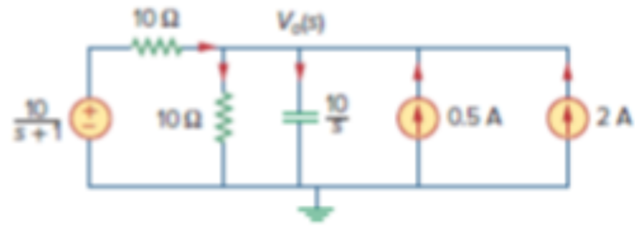
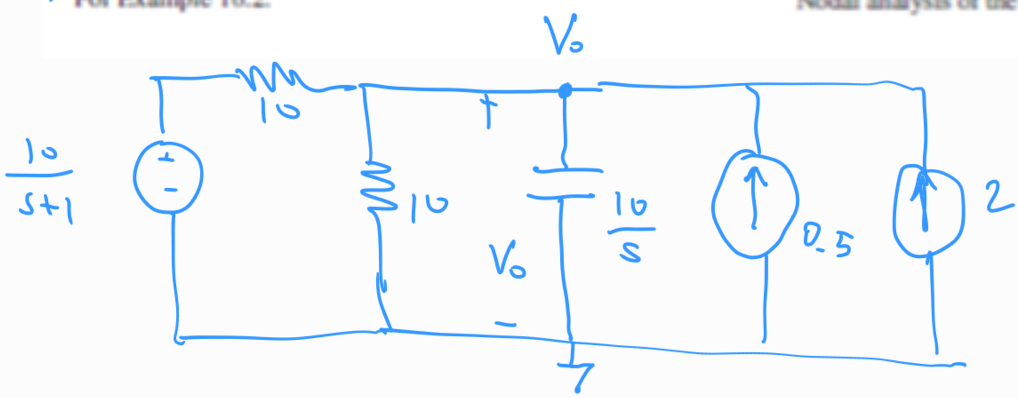


Figure 16.8
Nodal analysis of the equivalent of the circuit in Fig. 16.7.



$$i = C\frac{dv}{dt} \rightarrow I = sCV - Cv_o$$

$$\frac{V_o - \frac{10}{s+1}}{10} + \frac{V_o}{10} + \frac{V_o}{10/s} = 2.5$$

$$\frac{V_o}{10}\left(1 + 1 + s\right) = \frac{1}{s+1} + 2.5$$

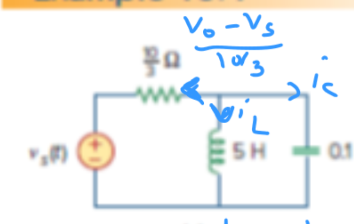
$$V_o = \frac{(2.5s + 3.5)}{(s+1)(s+2)} \times 10$$

$$= \frac{K_1^{\to 10}}{s+1} + \frac{K_2^{\to 15}}{s+2}$$

$$K_1 = \frac{1 \times 10}{1} = 10 \qquad K_2 = \frac{-1.5 \times 10}{-1} = 15$$

$$v_o(t) = \left(10e^{-t} + 15e^{-2t}\right) u(t)$$

Example 16.4



(a)

Consider the circuit in Fig. 16.12(a). Find the value of the voltage across the capacitor assuming that the value of $v_s(t) = 10u(t)$ V and assume that at $t = 0$, -1 A flows through the inductor and $+5$ V is across the capacitor

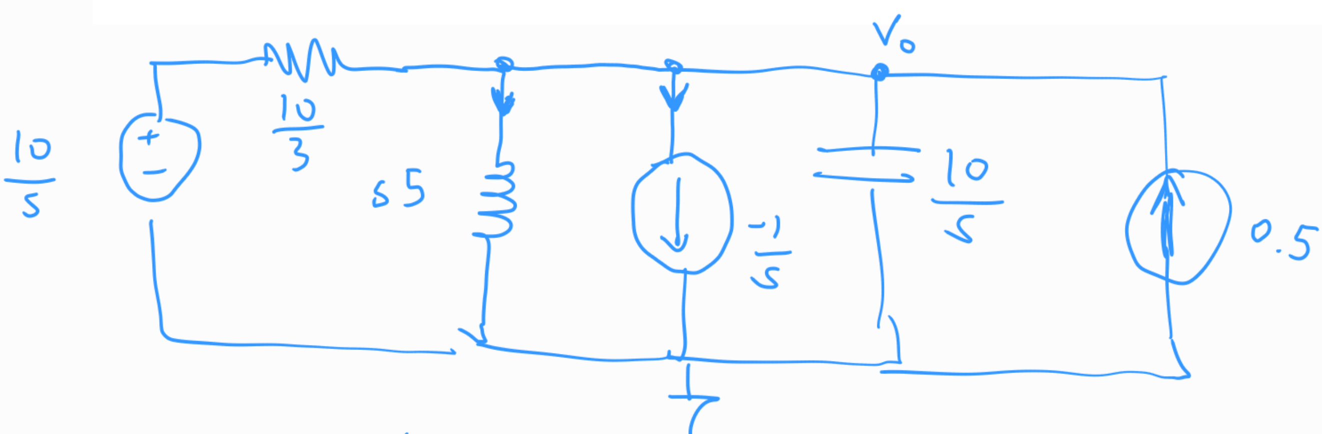
Solution:

Figure 16.12(b) represents the entire circuit in the s-domain with the initial conditions incorporated. We now have a straightforward nodal analysis problem. Because the value of V_1 is also the value of the capacitor voltage in the time domain and is the only unknown node voltage, we

Practice Problem 16.4

For the circuit shown in Fig. 16.12 with the same initial conditions, find the current through the inductor for all time $t > 0$.

Answer: $i(t) = (3 - 7e^{-t} + 3e^{-2t})u(t)$ A.



$$v = L i' = sLI - L i_0$$

$$\frac{V + L i_0}{sL} = \frac{V}{sL} + \frac{i_0}{s}$$

$$i = C v'$$

$$I = sCV - C v_0$$

$$\frac{V_o - \frac{10}{s}}{\frac{10}{3}} + \frac{V_o}{s5} - \frac{1}{s} + \frac{V_o s}{10} = 0{,}5$$

$$V_o\left(\frac{3}{10} + \frac{1}{5s} + \frac{s}{10}\right) = \underbrace{0.5 + \frac{1}{s}} + \frac{3}{s}$$

$$V_o\left(\frac{3s+s^2+2}{10\cancel{s}}\right) = \frac{0.5s+4}{\cancel{s}}$$

$$V_o = \frac{5s+40}{\underbrace{s^2+3s+2}_{(s+1)(s+2)}} = \frac{\overset{35}{K_1}}{s+1} + \frac{\overset{-30}{K_2}}{s+2}$$

$$v_o(t) = \left(35e^{-t} - 30e^{-2t}\right)u(t) \;//$$

$$\cancel{\frac{V_o - V_s}{20/3} + i_L + i_c = 0}$$

~~uzun yöntem~~

$$I_L(s) = \frac{V_o(s)}{5s} - \frac{1}{s} = \frac{5s+40}{5s(s+1)(s+2)} - \frac{1}{s}$$

$$= \frac{s+8}{s(s+1)(s+2)} - \frac{1}{s}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2} - \frac{1}{s}$$

$$= \frac{4}{s} - \frac{7}{s+1} + \frac{3}{s+2} - \frac{1}{s} = \frac{3}{s} - \frac{7}{s+1} + \frac{3}{s+2}$$

$$i_L(t) = \left(3 - 7e^{-t} + 3e^{-2t}\right)u(t) \;//$$

Example 16.6

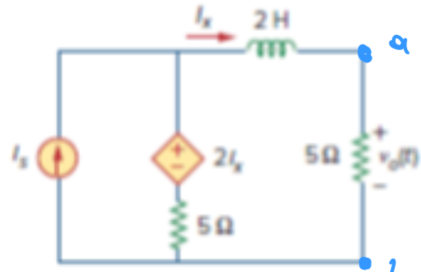


Figure 16.14
For Example 16.6.

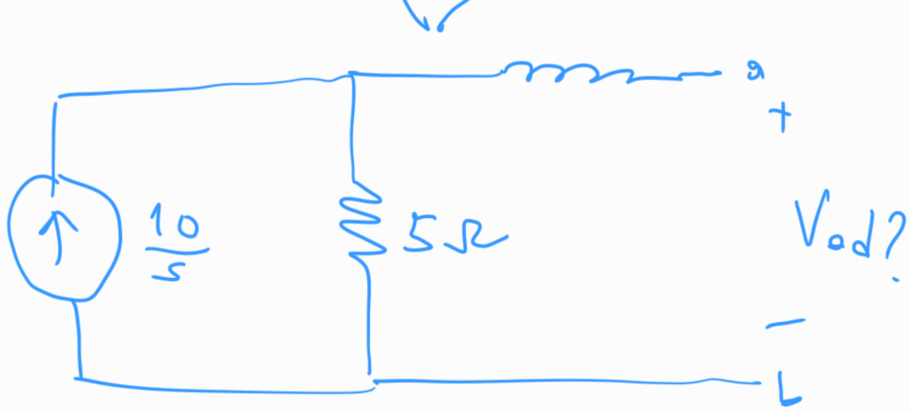
Assume that there is no initial energy stored in the circuit of Fig. 16.14 at $t = 0$ and that $i_s = 10u(t)$ A. (a) Find $V_o(s)$ using Thevenin's theorem. (b) Apply the initial- and final-value theorems to find $v_o(0^+)$ and $v_o(\infty)$. (c) Determine $v_o(t)$.

Solution:
Because there is no initial energy stored in the circuit, we assume that the initial inductor current and initial capacitor voltage are zero at $t = 0$.

(a) To find the Thevenin equivalent circuit, we remove the 5-Ω resistor and then find V_{oc} (V_{Th}) and I_{sc}. To find V_{Th}, we use the Laplace-transformed circuit in Fig. 16.15(a). Since $I_x = 0$, the dependent voltage source contributes nothing, so

a-b noktalarını açık devre yapalım. V_{ad} bulalım.

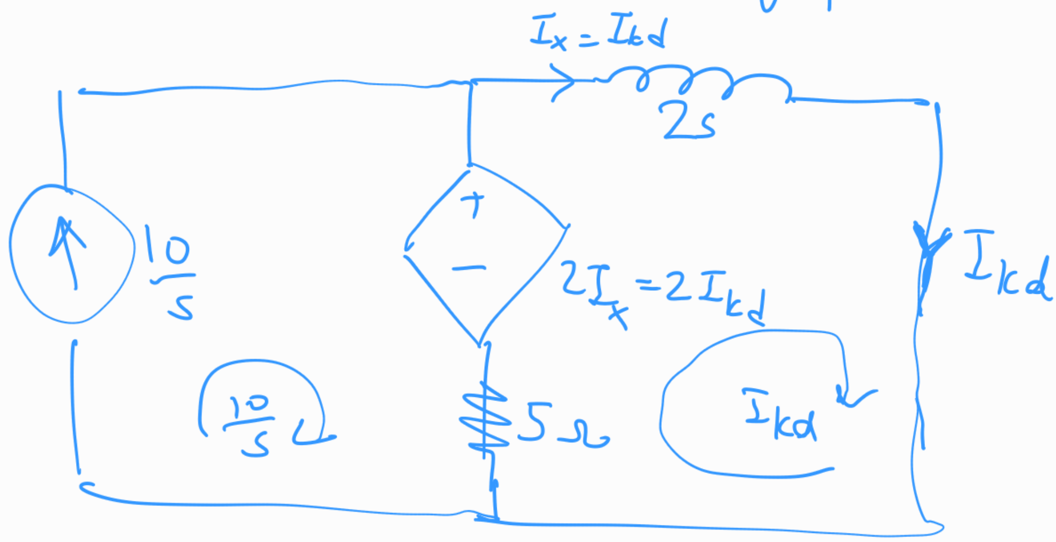
$I_x = 0$, $2I_x = 0$ etki etmez.



$\boxed{V_{ad} = \frac{50}{s}}$ ✓

↓ V_{Th}

a.b noktalarını kısa devre yapalım ve I_{kd}'yi bulalım

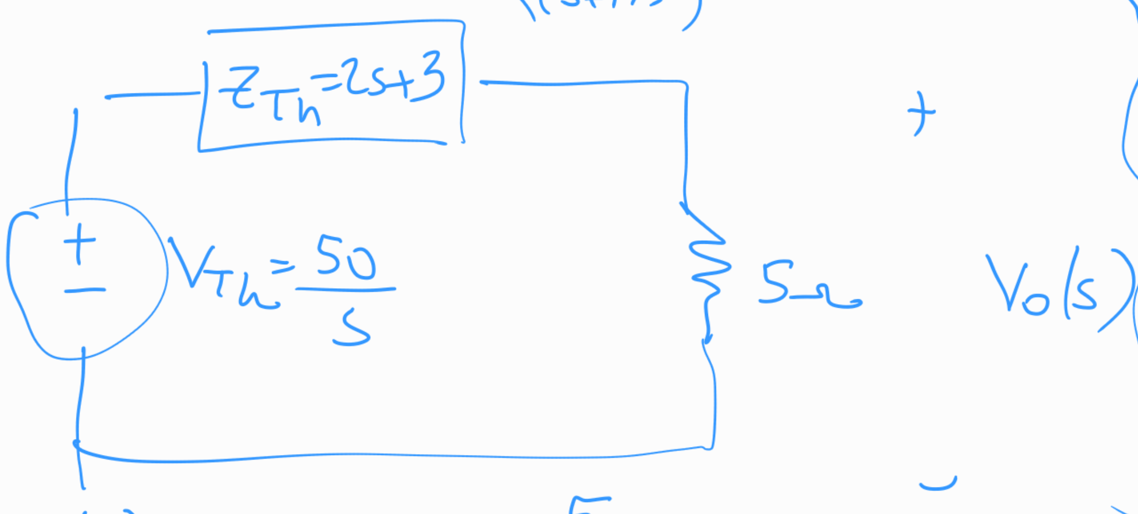


$$5\left(I_{kd} - \frac{10}{s}\right) - 2I_{kd} + 2s\,I_{kd} = 0 \rightarrow \text{S.A.Y.}$$

$$I_{kd}(5 - 2 + 2s) = \frac{50}{s}$$

$$I_{kd} = \frac{50}{s(2s+3)} = \frac{25}{s(s+1.5)}$$

$$\rightarrow Z_{Th} = \frac{V_{ad}}{I_{kd}} = \frac{\frac{50}{\cancel{s}}}{\frac{25}{\cancel{s}(s+1.5)}} = 2s+3$$



Thevenin eşdeğer devresi

$$V_o(s) = \frac{50}{s}\,\frac{5}{2s+3+5}$$

$$= \frac{125}{s(s+4)} = \frac{K_1}{s} + \frac{K_2}{s+4} = \frac{125/4}{s} - \frac{125/4}{s+4}$$

$$v_o(t) = \frac{125}{4}\left(1-e^{-4t}\right)u(t)$$

16.4 Transfer Fonksiyonu:

$$H(s) = \frac{Y(s)}{X(s)} \qquad \left(\text{ör. } \frac{V_o(s)}{V_{in}(s)}\right)$$

$$X(s)H(s) = Y(s)$$

$$H(s) \xrightarrow{\mathcal{L}^{-1}} h(t)$$ Dürtü tepkisi (impulse response)

$Y(s) = X(s)H(s) \xrightarrow{\mathcal{L}^{-1}} y(t) = x(t) * h(t)$

Convolution.

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

Transfer fonksiyonu hesabında başlangıç koşulları sıfır kabul edilir. Dikkate alınmaz.

Example 16.9

For the s-domain circuit in Fig. 16.19, find: (a) the transfer function $H(s) = V_o/V_i$, (b) the impulse response, (c) the response when $v_i(t) = u(t)$ V, (d) the response when $v_i(t) = 8\cos 2t$ V.

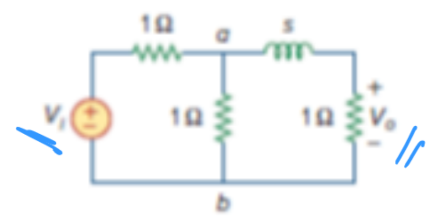


Figure 16.19 For Example 16.9.

Solution:

(a) Using voltage division,

$$V_o = \frac{1}{s+1} V_{ab} \qquad (16.9.1)$$

But

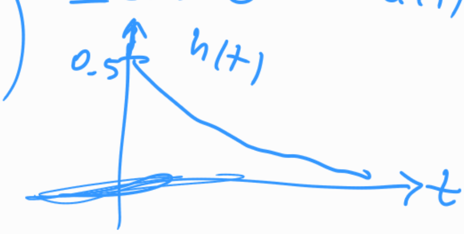
a)

$$V_o = V_i \frac{1 \,\|\, (s+1)}{1 + (1 \,\|\, (s+1))} \times \frac{1}{s+1}$$

$$\frac{V_o}{V_i} = \frac{\frac{\cancel{s+1}}{s+2}}{1 + \frac{s+1}{s+2}} \times \frac{1}{\cancel{s+1}} = \frac{1}{2s+3} = H(s)$$

b)

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left(\frac{1}{2s+3}\right) = 0.5e^{-1.5t}u(t)$$



c) $v_i(t) = u(t)$ ise $v_o(t)$?

① $v_o(t) = v_i(t) * h(t)$ ya da

② $V_o(s) = V_i(s)H(s)$

$V_i(s) = \frac{1}{s}$ $H(s) = \frac{1}{2s+3} \longrightarrow V_o(s) = \frac{1}{s(2s+3)}$

$= \frac{0.5}{s(s+1.5)}$

$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s+1.5}$ $\quad K_1 \to 1/3, \quad K_2 \to -1/3$

$v_o(t) = \frac{1}{3}\left(1 - e^{-1.5t}\right)u(t)$

$v_i(t) = 8\cos 2t$ ise $v_o(t)$

$V_o(s) = V_i(s)\, H(s)$

$= \frac{8s}{s^2+4} \frac{0.5}{s+1.5} = \frac{4s}{(s^2+4)(s+1.5)}$

$= \frac{K_1 s + K_2}{s^2+4} + \frac{K_3}{s+1.5}$ $\quad K_3 \to \frac{-24}{25}$

$K_3 = \frac{-6}{6.25}$

K_1, K_2?

$K_1 s^2 + 1.5 K_1 s + K_2 s + 1.5 K_2 - \frac{24}{25} s^2 - \frac{96}{25} = 4s$

$K_1 - \frac{24}{25} = 0 \Rightarrow K_1 = \frac{24}{25}$

$1.5 K_1 + K_2 = 4$

$\frac{36}{25} + K_2 = \frac{100}{25} \Rightarrow K_2 = \frac{-64}{25}$

$V_o(s) =$

$$\frac{\frac{24}{25}s - \frac{64}{25}}{s^2+4} - \frac{24/25}{s+1.5}$$

$$\frac{24}{25}\frac{s}{s^2+4} - \frac{32}{25} \times \frac{2}{s^2+4} - \frac{24/25}{s+1.5}$$

$$v_o(t) = \Big(\underbrace{\frac{24}{25}\cos(2t) - \frac{32}{25}\sin(2t)}_{\text{Yatışkın tepki (steady state)}} \underbrace{- \frac{24}{25} e^{-1,5t}}_{\text{Geçici (transient) tepki}}\Big) u(t)$$

$t \to \infty$: yatışkın (fazör ile de bulunabilir)