15.4 Ters Laplace Dönüşümü

$$F(s) = \frac{20}{(s+3)(s^2+8s+25)}$$

$$\rightarrow F(s) = \frac{2s(s^2-27)}{(s^2+9)^3}$$

Buradan tekrar zaman alanına nasıl dönülür?

- $F(s)$, iki polinomun bölümü
- Sıfırlar : Payı sıfır yapan s değerleri
- Kutuplar : Paydayı sıfır yapan s değerleri.
  - ↳ Paydayı çarpanlarına ayır.
    - ↳ $F(s)$'yi basit kesir terimlerine ayır.
      - ↳ ~~H~~er terimin ters Laplace dönüşümünü bul.

"Kısmi Kesir açılımı" "Partial fraction expansion"

15.4.1 Basit Kutuplar :

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$F(s) = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n} \rightarrow \text{Basit kesirler.}$$

$$F(s)\times(s+p_1) = K_1 + \frac{K_2(s+p_1)}{s+p_2} + \cdots + \frac{K_n(s+p_1)}{s+p_n}$$

$$F(s)\times(s+p_1)\Big|_{s=-p_1} = K_1$$

$$F(s)\times(s+p_k)\Big|_{s=-p_k} = K_k$$

$$f(t) = \left(K_1 e^{-p_1 t} + K_2 e^{-p_2 t} + \cdots + K_n e^{-p_n t}\right)u(t)$$

15.4.2 Tekrarlı Kutuplar:

$$F(s) = \frac{K_n}{(s+p)^n} + \frac{K_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{K_1}{s+p} + F_1(s)$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1} e^{-at}}{(n-1)!} u(t)$$

$$F(s)(s+p)^n = K_n + K_{n-1}(s+p) + \cdots + K_1 (s+p)^{n-1} + F_1(s)(s+p)^n$$

$$F(s)(s+p)^n \Big|_{s=-p} = K_n$$

$K_{n-1} = ?$

$$\left[\frac{d}{ds}\left\{F(s)(s+p)^n\right\}\right]_{s=-p} = K_{n-1}$$

$K_{n-2} = ?$

$$\left[\frac{d}{ds}\left\{F(s)(s+p)^n\right\}\right]_{s=-p} = 2K_{n-2}$$

$K_1 = ?$

$$\left[\frac{d}{ds}\left\{F(s)(s+p)^n\right\}\right]_{s=-p} = (n-1)!\,K_1$$

15.4.3 Karmaşık Kutuplar:

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)$$

$s^2 + as + b = (s+\alpha)^2 + \beta^2$ olarak yazarız.

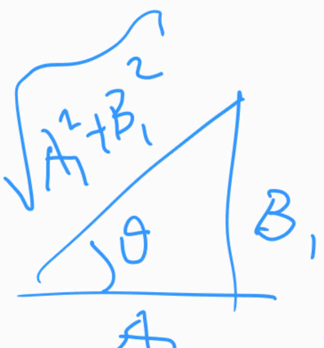
$$s^2 + 4s + 8 = (s+2)^2 + 2^2$$

$A_1 s + A_2 = A_1(s+\alpha) + B_1 \beta$ olarak yazarız

$$F(s) = \underbrace{\frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2}}_{\downarrow \mathcal{L}^{-1}} + \underbrace{\frac{B_1 \beta}{(s+\alpha)^2+\beta^2}}_{\downarrow \mathcal{L}^{-1}} + F_1(s)$$

$$f(t) = \underbrace{A_1 e^{-\alpha t}\cos(\beta t) + B_1 e^{-\alpha t}\sin\beta t}_{\text{sin ve cos birleştirilebilir}} + f_1(t)$$

$$\sqrt{A_1^2+B_1^2}\, e^{-\alpha t}\left[\underbrace{\frac{A_1}{\sqrt{A_1^2+B_1^2}}}_{\cos\theta}\cos(\beta t) + \underbrace{\frac{B_1}{\sqrt{A_1^2+B_1^2}}}_{\sin\theta}\sin\beta t\right]$$



$$f(t) = \sqrt{A_1^2+B_1^2}\, e^{-\alpha t}\cos(\beta t - \theta) + f_1(t)$$

Example 15.8

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

$$f(t) = \left(3 - 5e^{-t} + 3\sin(2t)\right)u(t)$$

Problem 15.8

$\cos(5t) \xrightarrow{\mathcal{L}} \frac{s}{s^2+25}$

$$F(s) = 5 + \frac{6}{s+4} - \frac{7s}{s^2+25}$$

$$5\delta(t) \quad 6e^{-4t}u(t) \quad -7\cos(5t)u(t)$$

$$f(t) = 5\delta(t) + \left(6e^{-4t} - 7\cos(5t)\right)u(t)$$

Example 15.9

$$F(s) = \frac{s^2+12}{s(s+2)(s+3)} = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$K_1 \to 2$, $K_2 \to -8$, $K_3 \to 2$

$$F(s)\times s\Big|_{s=0} = \frac{12}{2\times 3} = 2$$

$$F(s)\times(s+2)\Big|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$F(s)\times(s+3)\Big|_{s=-3} = \frac{9+12}{(-3)(-1)} = 7$$

$$f(t) = \left(2 - 8e^{-2t} + 7e^{-3t}\right)u(t)$$

Example: 15.10

$$V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2}$$

$$= \frac{K_1}{s+2} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s} + \frac{K_4}{s+1}$$

$K_1 \to 13$, $K_2 \to 22$, $K_3 \to 1$, $K_4 \to -14$

$$K_3 = \frac{4}{1 \times 2^2} = 1$$

$$K_4 = \frac{10+4}{(-1)(1)^2} = -14$$

$$K_2 = \frac{10(-2)^2+4}{(-2)(-1)} = \frac{44}{2} = 22$$

$$K_1 = \left[\frac{d}{ds}\left\{V(s)(s+2)^2\right\}\right]_{s=-2}$$

$$= \left[\frac{d}{ds}\frac{10s^2+4}{s(s+1)}\right]_{s=-2}$$

$$= \left.\frac{20s^2(s+1) - (10s^2+4)(2s+1)}{s^2(s+1)^2}\right|_{s=-2}$$

$$= \frac{-80 - (44)(-3)}{4 \times 1} = \frac{132-80}{4} = 13$$

$$v(t) = \left(13e^{-2t} + 22te^{-2t} + 1 - 14e^{-t}\right)u(t)$$

Example 15.11:

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)}$$

2

$$H(s) = \frac{K_1}{s+3} + \frac{K_2 s + K_3}{s^2+8s+25}$$

$K_2 \to -2$, $K_3 \to -10$

$$K_1 = \frac{20}{9-24+25} = 2$$

$$\underbrace{2s^2} + \overbrace{16s} + 50 + \underbrace{K_2 s^2} + \overbrace{K_3 s} + \overbrace{3K_2 s} + 3K_3 = 20$$

$$2 + K_2 = 0 \Rightarrow K_2 = -2$$

$$16 + K_3 + 3K_2 = 0$$

$$16 + K_3 - 6 = 0 \Rightarrow \boxed{K_3 = -10}$$

$$H(s) = \frac{2}{s+3} - \frac{2(s+5)}{\underbrace{s^2+8s+25}_{(s+4)^2+9}}$$

$$H(s) = \frac{2}{s+3} - 2 \cdot \left(\frac{s+4}{(s+4)^2+9} + \frac{1/3 \times 3}{(s+4)^2+9} \right)$$

$$h(t) = u(t)\left(2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin(3t)\right)$$

$$= \left(2e^{-3t} - \sqrt{\frac{40}{9}}\, e^{-4t}\cos\left(3t - \arctan\left(\frac{1}{3}\right)\right)\right)u(t)$$

Problem 15.11

$$G(s) = \frac{20}{(s+1)(s^2+4s+13)}$$

$$= \frac{K_1}{s+1} + \frac{K_2 s + K_3}{(s+2)^2+9}$$

$$K_1 = \frac{20}{1-4+13} = 2$$

$$K_1 s^2 + 4K_1 s + 13K_1 + K_2 s^2 + K_3 s + K_2 s + K_3 = 20$$

$$K_1 + K_2 = 0 \Rightarrow K_2 = -2$$

$$4K_1 + K_3 + K_2 = 0$$

$$8 + K_3 - 2 = 0 \Rightarrow K_3 = -6$$

$$G(s) = \frac{2}{s+1} - \frac{2(s+2) + \frac{2}{3} \cdot 3}{(s+2)^2 + 9}$$

$$g(t) = u(t)\left(2e^{-t} - 2 \cdot e^{-2t}\cos(3t) - \frac{2}{3}e^{-2t}\sin(3t)\right)$$

Başlangıç değeri:

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) = \underbrace{\int_{0^-}^{\infty} \frac{df}{\cancel{dt}} e^{-st} \cancel{dt}}_{s \to \infty\text{'a giderken } 0}$$

$$\lim_{s\to\infty} \left(sF(s) - f(0)\right) = 0$$

$$\boxed{\lim_{s\to\infty} sF(s) = f(0)}$$

$t = 0^-$'deki (başlangıç değeri)

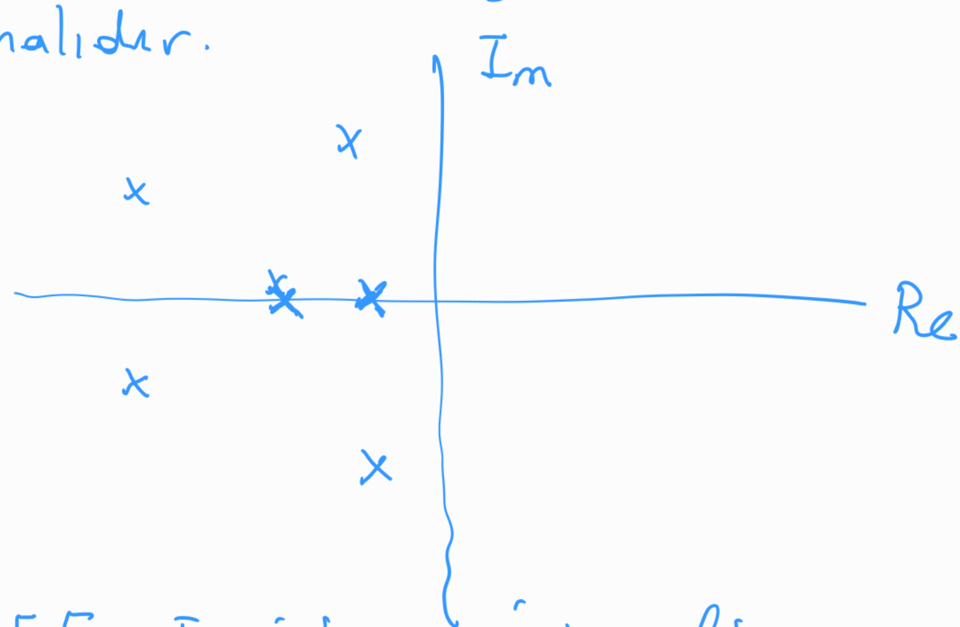
Son değer:

$$\lim_{s\to 0} \{\underbrace{sF(s) - \cancel{f(0^-)}}_{\text{türevin Laplace'i}}\} = \lim_{s\to 0} \int_0^{\infty} \frac{df}{dt} e^{-st} dt$$

$$= \int_0^{\infty} \frac{df}{\cancel{dt}} \cancel{dt} = f(\infty) - \cancel{f(0)}$$

$$\boxed{\lim_{s \to 0} \{sF(s)\} = f(\infty)}$$

Sistemin (devrenin) kararlı (stabil) olması için transfer fonksiyonunun kutupları hep sol yarı düzlemde (gerçel kısmı negatif) olmalıdır.



15.5 Evrişim integrali:

Bizim devrelerimiz (sistemimiz) → doğrusal (RLC → doğrusal, OPAMP → SAT'a girmediği sürece doğrusal)
→ zamanla değişmez.

$$\underbrace{Y(s)}_{\text{çıkış}} = \underbrace{X(s)}_{\text{girdi}} \underbrace{H(s)}_{\text{transfer fonk.}}$$

Zaman alanında:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{X(s)H(s)\} = x(t) * h(t)$$

evrişim (convolution)

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = y(t)$$

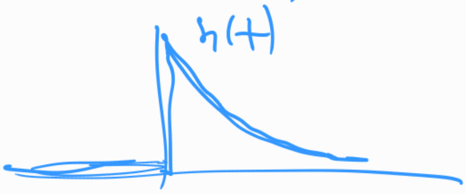
$x(\tau) \rightarrow$ girdi

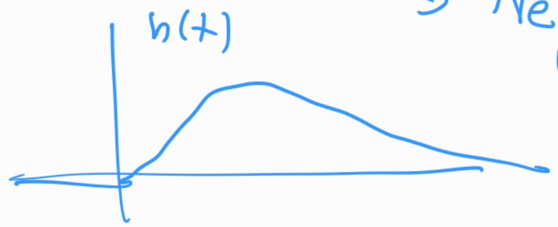
$h(t-\tau) \rightarrow$ dürtü tepkisi $\rightarrow$ impulse response

Pratikte:

$h(t) = 0$, $t < 0$ için $\longrightarrow$ Nedensel sistem

↳ causal





$h(t-\tau) = 0$, $t-\tau < 0$ için $\Rightarrow t < \tau$

$$x(t) * h(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

Genelde girdi $t=0$'da başlar.

$x(\tau) = 0$, $\tau < 0$ için

$$\boxed{x(t) * h(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau}$$

Özellikler:

1) Birleşme özelliği: $(x * y) * z = x * (y * z)$

2) Değişme $x * y = y * x$

3) Dağılma $(x+y) * z = x * z + y * z$

4) $\delta(t) * x(t) = \int_{0}^{t} \delta(\tau)\, x(t-\tau)\, d\tau$

$\delta(t) * x(t) = x(t)$

$$= \int_0^t \delta(\tau)\, x(t)\, d\tau$$

$$= x(t) \underbrace{\int_0^t \delta(\tau)\, d\tau}_{1}$$

$$= x(t)$$

5) $\delta(t-t_0) * x(t)$

$$= \int_0^t x(\tau)\, \delta(t-t_0-\tau)\, d\tau$$

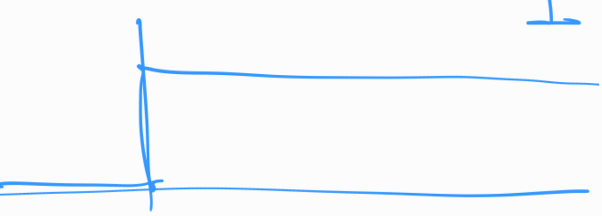
$$= \int_0^t x(t-t_0)\, \delta(t-t_0-\tau)\, d\tau$$

$$= x(t-t_0)$$

6) $\delta'(t) * x(t) \longrightarrow x'(t)$

7) $u(t) * x(t)$

$$\int_0^t \underbrace{u(\tau)}_{1}\, x(t-\tau)\, d\tau = \int_0^t x(t-\tau)\, d\tau$$



$=$ $t-\tau = u$

$d\tau = du$

$-d\tau = -du$

$t = \tau + u$

$$= \int_t^0 -x(u)\, du$$

$$= \boxed{\int_0^t x(u)\, du}$$

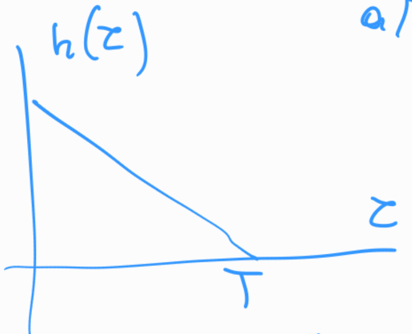
Evrişimin Grafiksel Olarak Alınması:

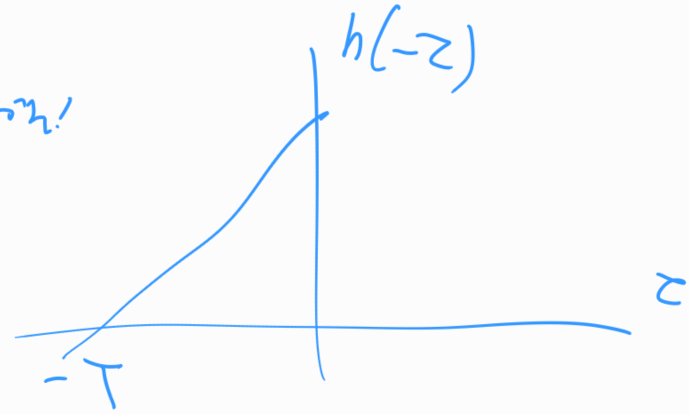
$$y(t) = x(t) * h(t) = \int_0^t x(\tau) * h(t-\tau)\, d\tau$$

$$= \int_0^t h(\tau)\, x(t-\tau)\, d\tau$$

$h(\tau) \rightarrow h(-\tau)$

→ dikey eksen etrafında simetriğini almak!



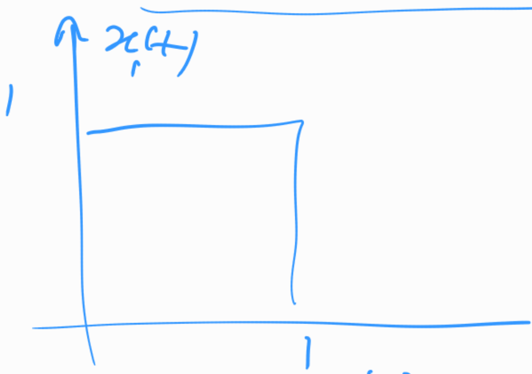


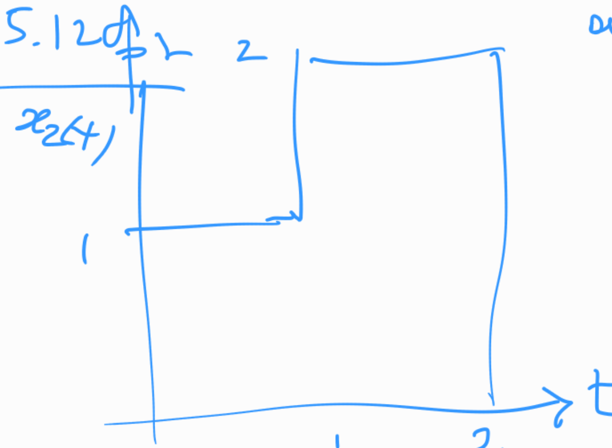
$h(-\tau) \rightarrow h(t-\tau)$

→ sağa t kadar kaydır.

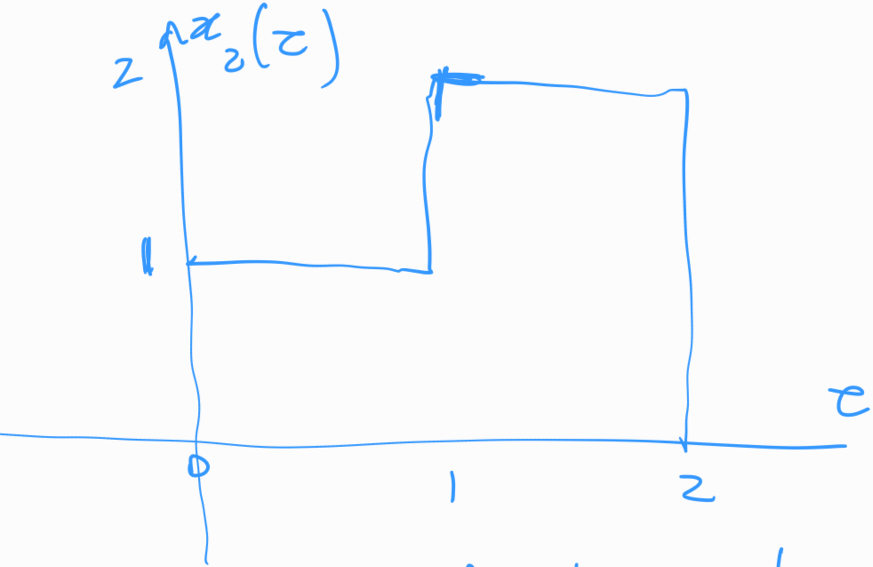
1) $h(\tau)$'nun dikey eksene göre simetriğini al
2) t kadar kaydırıp $x(\tau)$ ile çarp ve çarpımın $0-t$ arasında integralini al.

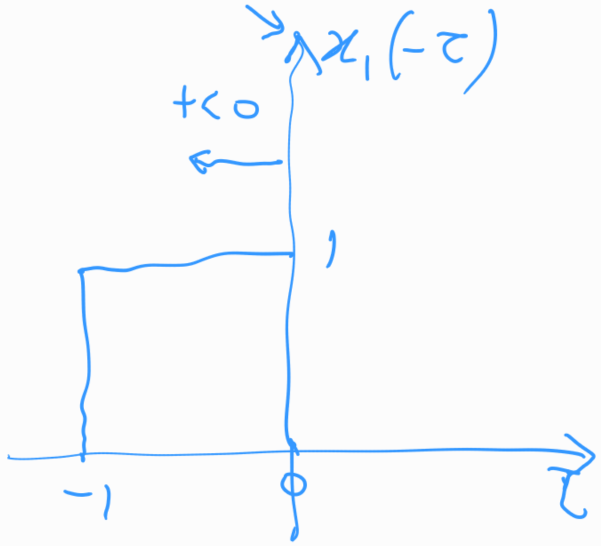
Problem 15.12





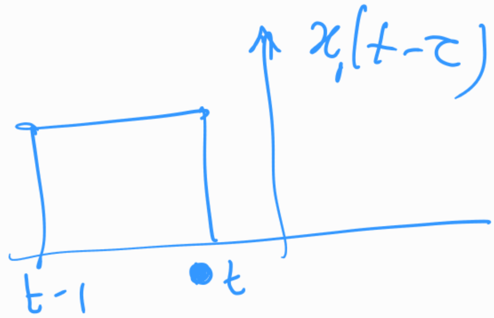
$$y(t) = x_1(t) * x_2(t) = ?$$





$\underline{t=0}$: hiç kaydırmadan $\longrightarrow y(0) = 0$

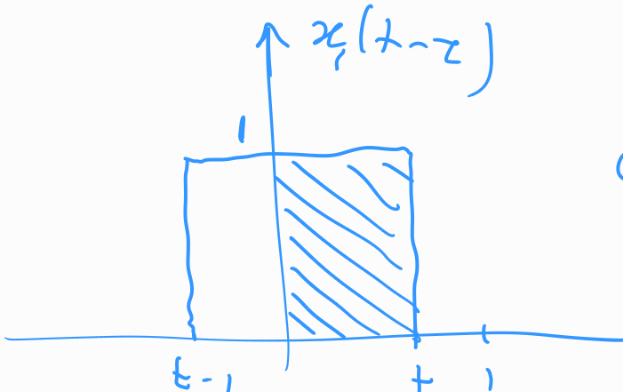
$\underline{t<0}$:



$\longrightarrow y(t) = 0 \quad t<0$

$\underline{0<t<1}$:

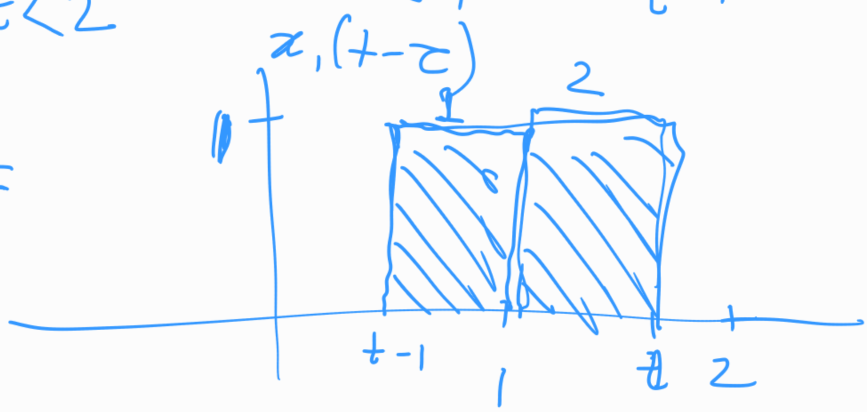
$y(t) = \underline{t}$

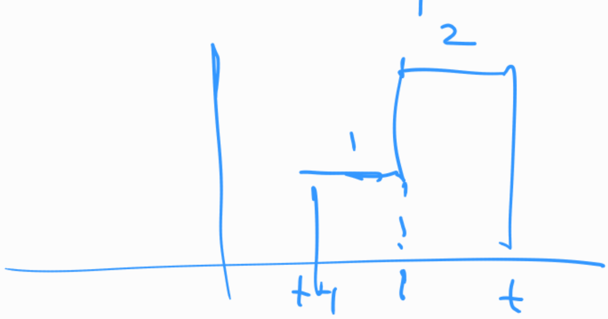


$$\underbrace{\int_0^t x_2(\tau)\, x_1(t-\tau)\, d\tau}_{t}$$

$1<t<2$

$\underline{y(t) = t}$



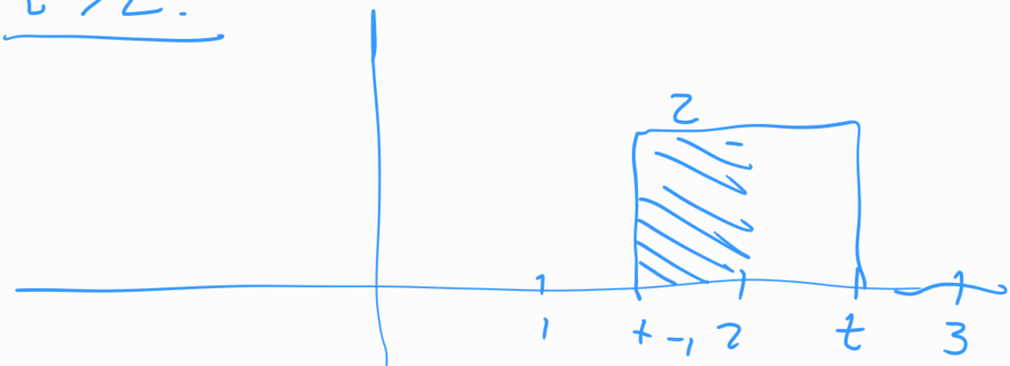


$(2-t)\times 1 + (t-1)2$

$\cancel{2}-t+2t-\cancel{2}$

t

$3 > t > 2$:



$$(2-t+1)\times 2 = 2(3-t)$$